On trees with equal domination and total outer-independent domination numbers

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Abstract

For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a total outer-independent dominating set if every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The domination (total outer-independent domination, respectively) number of G is the minimum cardinality of a dominating (total outer-independent dominating, respectively) set of G. We characterize all trees with equal domination and total outer-independent domination numbers.
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a tree H if there is a neighbor

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of v, say x, such that the tree resulting from T by removing the edge vx, and which contains the vertex x, is a tree H. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves.

A subset $D \subseteq V(G)$ is a dominating set, abbreviated DS, of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of minimum cardinality is called a $\gamma(G)$-set. For a comprehensive survey of domination in graphs, see [11, 12].

A subset $D \subseteq V(G)$ is a total dominating set of G if every vertex of G has a neighbor in D, while it is a total outer-independent dominating set, abbreviated TOIDS, of G if additionally the set $V(G) \setminus D$ is independent. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. A total outer-independent dominating set of G of minimum cardinality is called a $\gamma_t^{oi}(G)$-set. Total domination was introduced by Cockayne, Dawes, and Hedetniemi [5], and further studied for example [1–4, 6–10, 13, 14, 16, 17]. The study of total outer-independent domination in graphs was initiated in [15].

Trees with equal domination and total domination numbers were characterized in [17].

We characterize all trees with equal domination and total outer-independent domination numbers.

2 Results

Since the one-vertex graph does not have total outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following straightforward observations.

Observation 1 *Every support vertex of a graph G is in every $\gamma_t^{oi}(G)$-set.*

Observation 2 *For every graph G of diameter at least three there exists a $\gamma_t^{oi}(G)$-set that contains no leaf.*

Observation 3 *For every graph G of diameter at least two there exists a $\gamma(G)$-set that contains every support vertex.*

Observation 4 *For every graph G we have $\gamma_t^{oi}(G) \geq \gamma(G)$.*

We characterize all trees with equal domination and total outer-independent domination numbers. For this purpose we introduce a family $\mathcal{T}$ of trees

$T = T_k$ that can be obtained as follows. Let T_1 be a path P_4. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining its any vertex to a vertex of T_k adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_3$: Attach a path P_4 by joining one of the support vertices to a vertex of T_k which is not a leaf, and is adjacent to a support vertex.
- Operation $\mathcal{O}_4$: Attach a vertex by joining it to a leaf of T_k adjacent to a strong support vertex.
- Operation $\mathcal{O}_5$: Attach a double star with five vertices by joining one of the leaves adjacent to the strong support vertex to a vertex of T_k adjacent to a double star with five vertices through a leaf adjacent to the strong support vertex.
- Operation $\mathcal{O}_6$: Attach a double star with five vertices by joining one of the leaves adjacent to the strong support vertex to a vertex of T_k adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_7$: Attach a path P_4 by joining one of the support vertices to a leaf of T_k adjacent to a strong support vertex.

Now we prove that for every tree of the family $\mathcal{T}$, the domination and the total outer-independent domination numbers are equal.

Lemma 5 *If $T \in \mathcal{T}$, then $\gamma_t^{oi}(T) = \gamma(T)$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_4$, then obviously $\gamma_t^{oi}(T) = 2 = \gamma(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. The attached vertex we denote by x. Let y be its neighbor. Let D' be any $\gamma_t^{oi}(T')$-set. By Observation 1 we have $y \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')$. Now let D be a $\gamma(T)$-set that contains every support vertex. The set D is minimal, thus $x \notin D$. Obviously, D is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T)$. Now we get $\gamma_t^{oi}(T)$

$\leq \gamma_t^{oi}(T') = \gamma(T') \leq \gamma(T)$. On the other hand, by Observation 4 we have $\gamma_t^{oi}(T) \geq \gamma(T)$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which is attached P_2 we denote by x. Let v_1v_2 be the attached path. Let v_1 be joined to x. Let y be a support vertex of degree two adjacent to x and different from v_1. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex y has to have a neighbor in D', thus $x \in D'$. It is easy to see that $D' \cup \{v_1\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let D be a $\gamma(T)$-set that contains every support vertex. The set D is minimal, thus $v_2 \notin D$. Let us observe that $D \setminus \{v_1\}$ is a DS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1\}$. Therefore $\gamma(T') \leq \gamma(T) - 1$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1 = \gamma(T') + 1 \leq \gamma(T)$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. The vertex to which is attached P_4 we denote by x. Let $v_1v_2v_3v_4$ be the attached path. Let v_2 be joined to x. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{v_2, v_3\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let D be a $\gamma(T)$-set that contains every support vertex. The set D is minimal, thus $v_1, v_4 \notin D$. Let us observe that $D \setminus \{v_2, v_3\}$ is a DS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_2, v_3\}$. Therefore $\gamma(T') \leq \gamma(T) - 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 = \gamma(T') + 2 \leq \gamma(T)$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_4$. Let x be the attached vertex, and let y be its neighbor. The neighbor of y other than x we denote by z. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 1 we have $z \in D'$. It is easy to observe that $D' \cup \{y\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let D be a $\gamma(T)$-set that contains every support vertex. The set D is minimal, thus $x \notin D$. Let us observe that $D \setminus \{y\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1 = \gamma(T') + 1 \leq \gamma(T)$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_5$. The vertex to which is attached the double star we denote by x. Let v_1 be the leaf of the double star which is joined to x. The neighbor of v_1 other than x we denote by v_2. Let v_3 be the weak support vertex of the double star. The leaf adjacent to v_3 we denote by v_4. The remaining vertex of the double star we denote by v_5. Let u_1 be a neighbor of x which is a leaf of a double star with five vertices adjacent to the strong support vertex. The remaining vertices of this double star we denote similarly. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertices u_1, u_4, and u_5. Let D' be such a set. The set $V(T') \setminus D'$ is independent, thus $x \in D'$. It is easy to observe that $D' \cup \{v_2, v_3\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma(T)$-set that contains every support vertex, and does not contain the vertex v_1. Let D be such a set. The set D is minimal, thus $v_4, v_5 \notin D$. Observe that $D \setminus \{v_2, v_3\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 = \gamma(T) + 2 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_6$. The vertex to which is attached the double star we denote by x. Let v_1 be the leaf of the double star which is joined to x. The neighbor of v_1 other than x we denote by v_2. Let v_3 be the weak support vertex of the double star. The leaf adjacent to v_3 we denote by v_4. The remaining vertex of the double star we denote by v_5. Let y be a support vertex of degree two adjacent to x. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex y has to have a neighbor in D', thus $x \in D'$. It is easy to observe that $D' \cup \{v_2, v_3\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma(T)$-set that contains every support vertex, and does not contain the vertex v_1. Let D be such a set. The set D is minimal, thus $v_4, v_5 \notin D$. Observe that $D \setminus \{v_2, v_3\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 = \gamma(T') + 2 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_7$. Similarly as when considering the operation $\mathcal{O}_3$ we conclude that $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$ and $\gamma(T') \leq \gamma(T) - 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 = \gamma(T') + 2 \leq \gamma(T)$. This implies that $\gamma_t^{oi}(T) = \gamma(T)$. ■

Now we prove that if the domination and the total outer-independent domination numbers of a tree are equal, then the tree belongs to the family $\mathcal{T}$.

Lemma 6 *Let T be a tree. If $\gamma_t^{oi}(T) = \gamma(T)$, then $T \in \mathcal{T}$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $\gamma_t^{oi}(T) = 2 > 1 = \gamma(T)$. Now assume that $\mathrm{diam}(T) = 2$. Thus T is a star. We have $\gamma_t^{oi}(T) = 2 > 1 = \gamma(T)$.

Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a $\gamma(T')$-set that contains every support vertex. It is easy to see that D' is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. Obviously, D is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T)$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) = \gamma(T) \leq \gamma(T')$. On the other hand, by Observation 4 we have $\gamma_t^{oi}(T') \geq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. If $\mathrm{diam}(T) \geq 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geq 6$, then let e be the parent of d. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that $d_T(u) = 2$. First assume that there is a child of w other than u, say k, such that the distance of w to the most distant vertex of T_k is three. Let l be a support vertex which is a child of k. The leaf adjacent to l we denote by m. Let $T' = T - T_u$. Let D' be any $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 1 we have $v \in D$. Each one of the vertices v and l has to have a neighbor in D, thus $u, k \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex w has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma(T') - 1 < \gamma(T')$, a contradiction.

Now assume that there is a child of w, say k, such that the distance of w to the most distant vertex of T_k is two. Thus k is a support vertex. Let $T' = T - T_u$. Let D' be any $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 1 we have $v, k \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex w has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma(T') - 1 < \gamma(T')$, a contradiction.

Now assume that some child of w, say k, is a leaf. Let $T' = T - t$. Let D' be a $\gamma(T')$-set that contains every support vertex. The set D' is minimal, thus $v \notin D$. It is easy to observe that $D' \setminus \{u\} \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 1 we have $v, w \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. It is easy to observe that $D \setminus \{v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1 = \gamma(T) - 1 \leq \gamma(T') - 1 < \gamma(T')$, a contradiction.

If $d_T(w) = 1$, then $T = P_4 = T_1 \in \mathcal{T}$. Now assume that $d_T(w) = 2$. If $d_T(d) = 1$, then $T = P_5$. We have $\gamma_t^{oi}(T) = 3 > 2 = \gamma(T)$. Now assume that $d_T(d) \geq 2$. Let $T' = T - T_u$. Let D' be any $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertices t and w. Let D be such a set. By Observation 1 we have

$v \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. We have $d \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex w has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2$ $\leq \gamma(T') - 1 < \gamma(T')$, a contradiction.

Now assume that $d_T(u) \geq 3$. Assume that among the children of u there is a support vertex, say x, different from v. Let $T' = T - T_v$. Let D' be any $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 1 we have $v, x \in D$. Let us observe that $D \setminus \{v\}$ is a TOIDS of the tree T' as the vertex u has a neighbor in $D \setminus \{v\}$. Therefore $\gamma_t^{oi}(T')$ $\leq \gamma_t^{oi}(T) - 1$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1 = \gamma(T) - 1 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Thus we can assume that $d_T(u) = 3$ and the child of u different from v, say x, is a leaf. First assume that some child of w other than u, say k, is a support vertex. Let $T' = T - T_u$. Let D' be any $\gamma(T')$-set. It is easy to observe that $D' \cup \{u, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 1 we have $u, v, k \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex w has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Let us observe that we can assume that all children of w other than u are leaves. Let k be a leaf which is a child of w. Let $T' = T - t$. Let D' be a $\gamma(T')$-set that contains every support vertex. The set D' is minimal, thus $v \notin D'$. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 1 we have $v, u, w \in D$. Let us observe that $D \setminus \{v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1 = \gamma(T) - 1 \leq \gamma_t(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

If $d_T(w) = 1$, then let $T' = T - x$. We have $T' = P_4 = T_1 \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $d_T(w) = 2$. First assume that there is a child of d other than w, say k, such that the distance of d to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is isomorphic to T_w. Let $T' = T - T_w$. Let D' be any $\gamma(T')$-set. It is easy to observe that $D' \cup \{u, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex w, and does not contain any leaf. Let D be such a set. By Observation 1 we have $u, v \in D$. Observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_5$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibilities when T_k is isomorphic to T_u, or T_k is a path P_3. First assume that T_k is isomorphic to T_u. The child of k which is a support vertex we denote by l. The leaf adjacent to l we denote by m. Let $T' = T - m$. Let D' be a $\gamma(T')$-set that contains every support vertex. The set D' is minimal, thus $l \notin D'$. It is easy to see that $D' \cup \{l\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex w, and does not contain any leaf. Let D be such a set. By Observation 1 we have $l \in D$. The set $V(T) \setminus D$ is independent, thus $d \in D$. Let us observe that $D \setminus \{l\}$ is a TOIDS of the tree T' as the vertex k has a neighbor in $D \setminus \{l\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1 = \gamma(T) - 1 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that T_k is a path P_3, say klm. Let $T' = T - T_u - m$. Let D' be a $\gamma(T')$-set that contains every support vertex. We have $l \notin D'$ as the set D' is minimal. Let us observe that $D' \setminus \{k\} \cup \{l, u, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex w, and does not contain any leaf. Let D be such a set. By Observation 1 we have $u, v, l \in D$. The vertex l has to have a neighbor in D, thus $k \in D$. We have $d \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{u, v, l\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 3$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 3 = \gamma(T) - 3 \leq \gamma_t(T') - 1 < \gamma(T')$, a contradiction.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. Let $T' = T - T_w$. Let D' be any $\gamma(T')$-set. It is easy to observe that $D' \cup \{u, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex w, and does not contain any leaf. Let D be such a set. By Observation 1 we have $u, v \in D$. Observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_6$. Thus $T \in \mathcal{T}$.

Now assume that some child of d, say k, is a leaf. Let $T' = T - T_u$. Let D' be any $\gamma(T')$-set. It is easy to observe that $D' \cup \{u, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. Let D be any $\gamma_t^{oi}(T)$-set. By Observation 1 we have $d, u, v \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex w has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma(T')$. This implies that $\gamma_t^{oi}(T') = \gamma(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_7$. Thus $T \in \mathcal{T}$.

If $d_T(d) = 1$, then let $T' = T - T_v$. We have $T' = P_4 = T_1 \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. Now assume that $d_T(d) = 2$. Let T' be a tree obtained from $T - T_u$ by attaching a vertex, say y, and joining it to the vertex e. Let D' be a $\gamma(T')$-set that contains every support vertex. Let us observe that $D' \setminus \{d\} \cup \{u, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex w, and does not contain any leaf. Let D be such a set. By Observation 1 we have $u, v \in D$. The set $V(T) \setminus D$ is independent, thus $d \in D$. We have $e \in D$ as the vertex d has to be dominated. It is easy to observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T'. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we get $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2 = \gamma(T) - 2 \leq \gamma_t(T') - 1 < \gamma(T')$, a contradiction. ■

As an immediate consequence of Lemmas 5 and 6, we have the following characterization of the trees with equal domination and total outer-independent domination numbers.

Theorem 7 *Let T be a tree. Then $\gamma_t^{oi}(T) = \gamma(T)$ if and only if $T \in \mathcal{T}$.*

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