Total outer-independent domination in graphs

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Abstract

We initiate the study of total outer-independent domination in graphs. A total outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of a graph G is the minimum cardinality of a total outer-independent dominating set of G. First we discuss the basic properties of total outer-independent domination in graphs. We find the total outer-independent domination numbers for several classes of graphs. Next we prove lower and upper bounds on the total outer-independent domination number of a graph, and we characterize the extremal graphs. Then we study the influence of removing or adding vertices and edges. We also give Nordhaus-Gaddum type inequalities.
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1 Introduction

Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. The complement of G, denoted by $\bar{G}$, is a graph which has the same vertices as G, and in which two vertices are adjacent if and only if they are not adjacent in G. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex

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is a vertex adjacent to a leaf. We say that a vertex is isolated if it has no neighbor. Let $\delta(G)$ ($\Delta(G)$, respectively) mean the minimum (maximum, respectively) degree among all vertices of G. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. By $K_{p,q}$ we denote a complete bipartite graph with partite sets of cardinalities p and q. By a star we mean the graph $K_{1,m}$. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. By a subdivided star we mean a graph obtained from a star by subdividing each one of its edges. Generally, let $K_{t_1,t_2,\ldots,t_k}$ denote the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$, where $|S_i| = t_i$ for positive integers $i \leq t$. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. The clique number of G, denoted by $\omega(G)$, is the number of vertices of a greatest complete graph which is a subgraph of G.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a total dominating set of G if every vertex of G has a neighbor in D. The domination (total domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating (total dominating, respectively) set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3], and further studied for example in [1, 2, 4–8, 10–14]. For a comprehensive survey of domination in graphs, see [9].

A subset $D \subseteq V(G)$ is a total outer-independent dominating set, abbreviated TOIDS, of G if every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. A total outer-independent dominating set of G of minimum cardinality is called a $\gamma_t^{oi}(G)$-set.

We initiate the study of total outer-independent domination in graphs. First we discuss the basic properties of total outer-independent domination in graphs. We find the total outer-independent domination numbers for several classes of graphs. Next we prove some lower and upper bounds on the total outer-independent domination number of a graph, and we characterize the extremal graphs. Then we study the influence of removing or adding vertices and edges. We also give Nordhaus-Gaddum type inequalities.

2 Results

Since the one-vertex graph, as well as all graphs with an isolated vertex, does not have neither a total outer-independent dominating set nor a total dominating set, in this paper we consider only graphs without isolated vertices.

We begin with the following straightforward observations.

Observation 1 *For every graph G we have $\gamma_t^{oi}(G) \geq \gamma_t(G)$.*

Observation 2 *Every support vertex of a graph G is in every TOIDS of G.*

Observation 3 *For every connected graph G of diameter at least three there exists a $\gamma_t^{oi}(G)$-set that contains no leaf.*

Observation 4 *If $n \geq 2$ is an integer, then $\gamma_t^{oi}(K_n) = \max\{n-1, 2\}$.*

Observation 5 *For every integer $n \geq 2$ we have*

$$\gamma_t^{oi}(P_n) = \begin{cases} 2 & \text{if } n = 2; \\ \lfloor 2n/3 \rfloor & \text{if } n \geq 3. \end{cases}$$

Let us observe that for any non-negative integer there exists a graph such that the difference between its total outer-independent domination and total domination numbers equals that non-negative integer.

Observation 6 *For every integer $n \geq 3$ we have $\gamma_t^{oi}(K_n) = \gamma_t(K_n) + n - 3$.*

Observation 7 *If $n \geq 3$ is an integer, then $\gamma_t^{oi}(C_n) = \lfloor (2n+2)/3 \rfloor$.*

Observation 8 *For every integer $n \geq 4$ we have $\gamma_t^{oi}(W_n) = \lfloor n/2 \rfloor + 1$.*

Observation 9 *If p and q are positive integers, then $\gamma_t^{oi}(K_{p,q}) = \min\{p, q\} + 1$.*

Observation 10 *Let $k \geq 3$ be an integer, and let $t_1, t_2, \ldots, t_k$ be positive integers. Then $\gamma_t^{oi}(K_{t_1,t_2,\ldots,t_k}) = \sum_{i=1}^{k} t_i - \max\{t_1, t_2, \ldots, t_k\}$.*

Observation 11 *For every disjoint graphs $G_1, G_2, \ldots, G_k$ we have $\gamma_t^{oi}(G_1 \cup G_2 \cup \ldots \cup G_k) = \gamma_t^{oi}(G_1) + \gamma_t^{oi}(G_2) + \ldots + \gamma_t^{oi}(G_k)$.*

Since the complement of every total outer-independent dominating set is independent, we get the following lower bound on $\gamma_t^{oi}(G)$ for any graph G.

Observation 12 *For every graph G we have $\gamma_t^{oi}(G) \geq n - \alpha(G)$.*

We have the following lower bound on the total outer-independent domination number of a graph in terms of its clique number.

Fact 13 *For every graph G we have $\gamma_t^{oi}(G) \geq \omega(G) - 1$.*

Proof. Let D be a $\gamma_t^{oi}(G)$-set, and let A be a maximum clique in G. Since $V(G) \setminus D$ is independent, we have $|(V(G) \setminus D) \cap A| \leq 1$. This implies that $|D| \geq |A| - 1$. We now get $\gamma_t^{oi}(G) = |D| \geq |A| - 1 = \omega(G) - 1$. ■

Let us observe that the bound from the previous proposition is tight. For $n \geq 3$ we have $\gamma_t^{oi}(K_n) = n - 1 = \omega(K_n) - 1$.

Now let us observe that for any non-negative integer there exists a graph such that the difference between its total outer-independent domination number and clique number equals that non-negative integer.

Observation 14 *Let G be a subdivided star. We have $\gamma_t^{oi}(G) = \omega(G) + \Delta(G) - 1$.*

We now prove that the total outer-independent domination number of a graph is greater than or equal to the minimum degree among all its vertices.

Fact 15 *For every graph G we have $\gamma_t^{oi}(G) \geq \delta(G)$.*

Proof. Let D be any $\gamma_t^{oi}(G)$-set. If $D = V(G)$, then obviously the result is true. Now assume that $D \neq V(G)$. Let x be a vertex which does not belong to D. Since $V(G) \setminus D$ is independent, all neighbors of x belong to the set D. Thus $|D| \geq d_G(x)$. By the definition we have $\delta(G) \leq d_G(x)$. Therefore $\gamma_t^{oi}(G) = |D| \geq d_G(x) \geq \delta(G)$. ■

Since every total outer-independent dominating set has at least two vertices, and all vertices of a graph form a total outer-independent dominating set, we have the following bounds on the total outer-independent domination number of a graph.

Observation 16 *For every graph G we have $2 \leq \gamma_t^{oi}(G) \leq n$.*

We now characterize all graphs which attain the lower bound from the previous observation. For this purpose we introduce a family $\mathcal{G} = \{G_{k,l,m} \colon k, l, m$ are non-negative integers$\}$ of graphs, an element of which is given in Figure 1. A graph $G_{k,l,m}$ has $k + l + m + 2$ vertices.

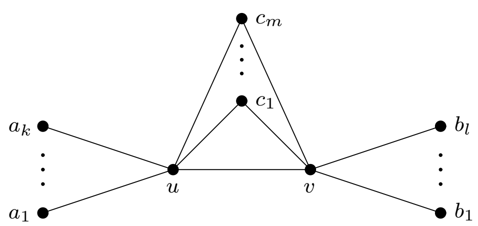

Figure 1: A graph $G_{k,l,m}$ of the family $\mathcal{G}$

Theorem 17 *Let G be a graph. We have $\gamma_t^{oi}(G) = 2$ if and only if $G \in \mathcal{G}$.*

Proof. It is easy to see that $\{u, v\}$ is a TOIDS of any graph of the family $\mathcal{G}$. This implies that $\gamma_t^{oi}(G) = 2$, for every graph G of the family $\mathcal{G}$. Now assume that for some graph G we have $\gamma_t^{oi}(G) = 2$. Let D be a $\gamma_t^{oi}(G)$-set. The vertices of D we denote by u and v. Since D is outer-independent, there is no edge between any two vertices of $V(G) \setminus D$. The set D is total dominating, thus every vertex of $V(G) \setminus D$ is adjacent to u or v. In particular, the vertices u and v are adjacent. Clearly, each vertex of $V(G) \setminus D$ is adjacent to at least one of the vertices u and v. It is easy to observe that G is a graph of the form presented in Figure 1. ∎

We now characterize the graphs attaining the upper bound from Observation 16, that is, the graphs with total outer-independent domination number equaling the number of vertices.

Theorem 18 *Let G be a graph. We have $\gamma_t^{oi}(G) = n$ if and only if $G = mK_2$.*

Proof. We have $\gamma_t^{oi}(mK_2) = m \cdot \gamma_t^{oi}(K_2) = 2m = n$. Now assume that for some graph G we have $\gamma_t^{oi}(G) = n$. Let H be any connected component of G. The graph G has no isolated vertices, thus $|V(H)| \geq 2$. If $|V(H)| \geq 3$, then some vertex of H, say x, is not a support vertex. Thus x is not adjacent to any leaf. Let us observe that $V(G) \setminus \{x\}$ is a TOIDS of the graph G. Therefore $\gamma_t^{oi}(G) \leq n-1 < n$, a contradiction. This implies that $H = K_2$, and consequently, $G = mK_2$. ∎

Corollary 19 *Let G be a graph. If $G \neq mK_2$, then $\gamma_t^{oi}(G) \leq n - 1$.*

We now characterize connected graphs attaining the bound from the previous corollary.

Theorem 20 *Let G be a connected graph. We have $\gamma_t^{oi}(G) = n - 1$ if and only if $G \in \{P_3, C_4, C_5\}$ or G is a complete graph on at least three vertices.*

Proof. We have $\gamma_t^{oi}(P_3) = 2 = 3 - 1$, $\gamma_t^{oi}(C_4) = 3 = 4 - 1$ and $\gamma_t^{oi}(C_5) = 4 = 5 - 1$. If $n \geq 3$, then by Observation 4 we have $\gamma_t^{oi}(K_n) = n - 1$. Now assume that for some graph G we have $\gamma_t^{oi}(G) = n - 1$. If $n = 2$, then $G = K_2$ and $\gamma_t^{oi}(G) = 2 = n > n - 1$, a contradiction. If $n = 3$, then $G = P_3$ or $G = K_3$. Now assume that $n \geq 4$. If $\mathrm{diam}(G) = 1$, then G is a complete graph. Now assume that $\mathrm{diam}(G) \geq 2$. Let us observe that there exist two nonadjacent vertices of G, say x and y, such that no common neighbor of x and y has degree two. It is not very difficult to observe that $V(G) \setminus \{x, y\}$ is a TOIDS of the graph G. Therefore $\gamma_t^{oi}(G) \leq n - 2 < n - 1$, a contradiction. ∎

As a corollary we give a characterization of all graphs with the total outer-independent domination number equaling the number of vertices minus one.

Corollary 21 *Let G be a graph. We have $\gamma_t^{oi}(G) = n - 1$ if and only if exactly one component of G, say H, is such that $H \in \{P_3, C_4, C_5\}$ or H is a complete graph on at least three vertices, while every other component of G is isomorphic to K_2.*

We have the following upper bound on the total outer-independent domination number of a graph in terms of its diameter.

Proposition 22 *If G is a graph of diameter d, then $\gamma_t^{oi}(G) \leq n - \lceil (d+1)/3 \rceil$.*

Proof. Let $v_0, v_1, ..., v_d$ be a diametrical path in G. Let G' be a graph obtained from G by removing all leaves adjacent any of the vertices $v_2, v_3, ..., v_{d-2}$. It is not difficult to observe that $\gamma_t^{oi}(G) \leq \gamma_t^{oi}(G') + |V(G) \setminus V(G')|$. Notice that $\mathrm{diam}(G') = \mathrm{diam}(G)$. If $d = 3k$, then let $S = \{v_{3i} \colon 0 \leq i \leq d/3\}$. If $d = 3k+1$, then let $S = \{v_0\} \cup \{v_{3i+1} \colon 1 \leq i \leq (d-1)/3\}$. If $d = 3k+2$, then let $S = \{v_{3i} \colon 0 \leq i \leq (d-2)/3\}$. Let us observe that $V(G') \setminus S$ is a TOIDS of the graph G'. We now get $\gamma_t^{oi}(G) \leq \gamma_t^{oi}(G') + |V(G) \setminus V(G')| \leq |V(G') \setminus S| + |V(G) \setminus V(G')| = n - \lceil (\mathrm{diam}(G') + 1)/3 \rceil = n - \lceil (d+1)/3 \rceil$. ■

Let us observe that the bound from the previous proposition is tight. We have $\gamma_t^{oi}(P_n) = \lfloor 2n/3 \rfloor = n - \lfloor (n+2)/3 \rfloor = n - \lfloor (d+3)/3 \rfloor = n - \lceil (d+1)/3 \rceil$.

We have the following upper bound on the total outer-independent domination number of a tree in terms of the maximum degree among all its vertices.

Lemma 23 *For every tree T we have $\gamma_t^{oi}(T) \leq n - \Delta(T) + 1$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $\gamma_t^{oi}(P_2) = 2 = n - \Delta(P_2) + 1$. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,m}$. We have $\Delta(K_{1,m}) = m$, $\gamma_t^{oi}(K_{1,m}) = 2$, and consequently, $\gamma_t^{oi}(K_{1,m}) = 2 = m + 1 - m + 1 = n - \Delta(K_{1,m}) + 1$. If $\mathrm{diam}(T) = 3$, then T is a double star. We have $\gamma_t^{oi}(T) = 2$, $\Delta(T) \leq n - 2$, and consequently, $\gamma_t^{oi}(T) = 2 \leq n - \Delta(T)$. Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order n of the tree T is at least five. We obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$. Let x be a vertex of T of maximum degree. Since T is not a star, there exists a leaf, say y, which is not adjacent to x. The neighbor of y we denote by z. Let $T' = T - y$. We have $n' = n - 1$ and $\Delta(T') = \Delta(T)$. Let D' be any $\gamma_t^{oi}(T')$-set. If $z \in D'$, then it is easy to see that D' is a TOIDS of the tree T. Now assume that $z \notin D'$. Observe that $D' \cup \{z\}$ is a TOIDS of the tree T. Therefore $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. We now get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1 \leq n' - \Delta(T') + 2 = n - \Delta(T) + 1$. ■

The next fact follows from the proof of the previous lemma.

Fact 24 *Let T be a tree. We have $\gamma_t^{oi}(T) = n - \Delta(T) + 1$ if and only if T has diameter at most two.*

We now easily get the following corollary.

Corollary 25 *For every tree T of diameter at least three we have $\gamma_t^{oi}(T) \leq n - \Delta(T)$.*

By subdividing an edge at most twice we mean subdividing it once, twice, or not subdividing it at all. We now characterize all trees attaining the bound from Corollary 25. For this purpose we introduce a family $\mathcal{T}$ of trees that can be obtained from a star either by subdividing one of its edges thrice, or by subdividing every edge at most twice in such way that some edge is subdivided at least once and some edge is subdivided at most once.

Theorem 26 *Let T be a tree. We have $\gamma_t^{oi}(T) = n - \Delta(T)$ if and only if T belongs to the family $\mathcal{T}$.*

Proof. Let T be a tree of the family $\mathcal{T}$. If T was obtained from $K_{1,2}$, then $T \in \{P_4, P_5, P_6\}$. We have $\Delta(P_4) = \Delta(P_5) = \Delta(P_6) = 2$, $\gamma_t^{oi}(P_4) = 2$, $\gamma_t^{oi}(P_5) = 3$ and $\gamma_t^{oi}(P_6) = 4$. We now get $\gamma_t^{oi}(P_4) = 2 = |V(P_4)| - \Delta(P_4)$, $\gamma_t^{oi}(P_5) = 3 = |V(P_5)| - \Delta(P_5)$ and $\gamma_t^{oi}(P_6) = 4 = |V(P_6)| - \Delta(P_6)$. Now assume that T was obtained from $K_{1,m}$ for some $m \geq 3$. The vertex of T of maximum degree we denote by x. If some edge of $K_{1,m}$ was subdivided thrice, then observe that all vertices which are not leaves form a $\gamma_t^{oi}(T)$-set. Thus $\gamma_t^{oi}(T) = n - \Delta(T)$. Now assume that while obtaining T from a star, every edge was subdivided at most twice in such way that some edge was subdivided at least once and some edge was subdivided at most once. Notice that T has diameter at least three. Let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. Since some edge was subdivided at most once, the vertex x is adjacent to a leaf or to a support vertex of degree two. Let us observe that all vertices of T which are not leaves belong to the set D. We now get $\gamma_t^{oi}(T) = |D| = n - \Delta(T)$.

Now assume that for some tree T we have $\gamma_t^{oi}(T) = n - \Delta(T)$. Fact 24 implies that $\text{diam}(T) \geq 3$. Let x be the vertex of T of maximum degree. Clearly, T has at least $\Delta(T)$ leaves. Suppose that it has more than $\Delta(T)$ leaves. By Observation 3, there exists a $\gamma_t^{oi}(T)$-set that contains no leaf. Therefore $\gamma_t^{oi}(T) \leq n - \Delta(T) - 1 < n - \Delta(T)$, a contradiction. Thus the tree T has exactly $\Delta(T)$ leaves. Let us observe that T can be obtained from a star by multiple subdividing its edges. Assume that while obtaining the tree T, some edge of a star was subdivided at least thrice. Thus there is a neighbor of x, say y, such that the tree resulting from T by removing the edge xy, and which contains the vertex y, is a path on at least four vertices. Let a be the vertex of this path which is a leaf of T. The neighbor of a we denote by b, and the neighbor of b other than a we denote by c. Let $T' = T - a - b - c$. Obviously, $\Delta(T') = \Delta(T)$. If $\text{diam}(T') = 2$, then T can be obtained from a star by subdividing one of its edges thrice. Now assume that $\text{diam}(T') \geq 3$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{b, c\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Using Corollary 25 we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')+2 \leq |V(T')|-\Delta(T')+2 = |V(T)|-3-\Delta(T)+2 = n-\Delta(T)-1 < n-\Delta(T)$, a contradiction. Therefore the tree T can be obtained from a star by subdividing every edge at most twice. Suppose that while obtaining the tree T every edge of a star was subdivided exactly twice. Obviously, $n = 3\Delta(T)+1$. Let us observe that $\gamma_t^{oi}(T) = 2\Delta(T)$. We now get $\gamma_t^{oi}(T) = 2\Delta(T) = 3\Delta(T)+1 -\Delta(T)-1 = n-\Delta(T)-1 < n-\Delta(T)$, a contradiction. This implies that while obtaining the tree T some edge of a star was subdivided at most once. Obviously, to obtain the tree T we have to subdivide some edge of a star $K_{1,m}$, otherwise $T = K_{1,m}$ and $\gamma_t^{oi}(T) = 2 > 1 = m+1-m = n-\Delta(T)$. ■

We have the following bounds on the total outer-independent domination number of a graph in terms of its order and size.

Proposition 27 *For every graph G we have*

$$\frac{2n-3-\sqrt{(2n-3)^2-8(m-1)}}{2} \leq \gamma_t^{oi}(G) \leq \frac{2n-3+\sqrt{(2n-3)^2-8(m-1)}}{2}.$$

Proof. Let D be a $\gamma_t^{oi}(G)$-set. Let t denote the number of edges between the vertices of D and the vertices of $V(G)\setminus D$. We have $m \leq t+|E(G[D])|$. Obviously, $t \leq |D| \cdot |V(G) \setminus D|$. Notice that $|E(G[D])| \leq (|D|-1)(|D|-2)/2$. Now simple calculations imply the result. ■

We now study the influence of the removal of a vertex of a graph on its total outer-independent domination number.

Proposition 28 *Let G be a graph, and let v be a vertex of G. If $G-v$ has no isolated vertex, then $\gamma_t^{oi}(G)-2 \leq \gamma_t^{oi}(G-v) \leq \gamma_t^{oi}(G)+d_G(v)-1$.*

Proof. Let D be any $\gamma_t^{oi}(G)$-set. If $v \notin D$, then it is easy to see that D is a TOIDS of the graph $G-v$. Now assume that $v \in D$. Let $v_1, v_2, \dots, v_{d_G(v)}$ be the neighbors of v. Let x_i be a neighbor of v_i other than v, for $i \in \{1,2,\dots,d_G(v)\}$. Let us observe that $D\cup\{x_1,x_2,\dots,x_{d_G(v)}\}\setminus\{v\}$ is a TOIDS of the graph G. Thus $\gamma_t^{oi}(G-v) \leq |D\cup\{x_1,x_2,\dots,x_{d_G(v)}\}\setminus\{v\}| \leq |D|-1+d_G(v) = \gamma_t^{oi}(G)+d_G(v)-1$. Now let D' be any $\gamma_t^{oi}(G-v)$-set. If some vertex of $N_G(v)$ belongs to the set D', then it is easy to see that $D' \cup \{v\}$ is a TOIDS of the graph G. Now assume that no vertex of $N_G(v)$ belongs to the set D'. Let x be any vertex of $N_G(v)$. It is easy to observe that $D' \cup \{v,x\}$ is a TOIDS of the graph G. Therefore $\gamma_t^{oi}(G) \leq \gamma_t^{oi}(G-v)+2$. ■

Let us observe that the bounds from the previous proposition are tight. For the lower bound, let G be a graph obtained from a star by subdividing every edge thrice. Observe that $\gamma_t^{oi}(G) = 2\Delta(G)+2$. The vertex of G which has minimum eccentricity we denote by v. Obviously, $G-v = \Delta(G)P_4$. Consequently, $\gamma_t^{oi}(G-v) = \gamma_t^{oi}(\Delta(G)P_4) = \Delta(G) \cdot \gamma_t^{oi}(P_4) = 2\Delta(G)$. We now get $\gamma_t^{oi}(G) = 2\Delta(G)+2 = \gamma_t^{oi}(G-v)+2$. For the upper bound, let G be a subdivided star. It is easy to observe that $\gamma_t^{oi}(G) = \Delta(G)+1$. The vertex of minimum eccentricity we denote by v. Obviously, $d_G(v) = \Delta(G)$. We have $G-v = \Delta(G)K_2$. Consequently, $\gamma_t^{oi}(G-v) = \gamma_t^{oi}(\Delta(G)K_2) = \Delta(G) \cdot \gamma_t^{oi}(K_2) = 2\Delta(G) = \Delta(G)+1+\Delta(G)-1 = \gamma_t^{oi}(G) + d_G(v) - 1$.

We now show that for any non-negative integer there exists a graph such that the removal of some its vertex increases the total outer-independent domination number by that non-negative integer.

Fact 29 *For every non-negative integer k there exists a graph G such that for some vertex v of G we have $\gamma_t^{oi}(G-v) - \gamma_t^{oi}(G) = k$.*

Proof. Let $m = k+1$, and let G be a subdivided star with m leaves. Let v be the vertex of G of minimum eccentricity. It is easy to see that $\gamma_t^{oi}(G) = m+1$. We have $G-v = mK_2$. Obviously, $\gamma_t^{oi}(mK_2) = m \cdot \gamma_t^{oi}(K_2) = 2m$. We now get $\gamma_t^{oi}(G-v) - \gamma_t^{oi}(G) = 2m-(m+1) = m-1 = k$. ∎

We now study the influence of the removal of an edge of a graph on its total outer-independent domination number.

Proposition 30 *Let G be a graph, and let e be an edge of G. If $G-e$ has no isolated vertex, then $\gamma_t^{oi}(G) - 1 \le \gamma_t^{oi}(G-e) \le \gamma_t^{oi}(G) + 2$.*

Proof. Let $e = xy$ and let D be a $\gamma_t^{oi}(G)$-set. Let a denote a neighbor of x other than y, and let b denote a neighbor of y other than x. It is easy to observe that $D \cup \{a,b\}$ is a TOIDS of the graph $G-e$. Thus $\gamma_t^{oi}(G-e) \le |D \cup \{a,b\}| \le |D|+2 = \gamma_t^{oi}(G)+2$. Now let D' be a $\gamma_t^{oi}(G-e)$-set. If some of the vertices x and y belongs to the set D', then D' is a TOIDS of the graph G. If none of the vertices x and y belongs to the set D', then it is easy to observe that $D' \cup \{x\}$ is a TOIDS of the graph G. Therefore $\gamma_t^{oi}(G) \le \gamma_t^{oi}(G-e)+1$. ∎

Let us observe that the bounds following from the previous proposition are tight. For the lower bound, let us remove an edge of the complete graph K_4. For the upper bound, consider a path P_4 and its central edge.

Similarly, adding an edge has the following influence on the total outer-independent domination number of a graph.

Proposition 31 *Let G be a graph. If $e \notin E(G)$, then $\gamma_t^{oi}(G) - 2 \le \gamma_t^{oi}(G+e) \le \gamma_t^{oi}(G) + 1$.*

We now give Nordhaus-Gaddum type inequalities for the sum of the total outer-independent domination number of a graph and its complement.

Theorem 32 *For every graph G we have $n-1 \leq \gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) \leq 2n-1$.*

Proof. Let D be any $\gamma_t^{oi}(G)$-set. Since $V(G) \setminus D$ is independent, the vertices of $V(G) \setminus D$ form a clique in $\bar{G}$. Let $\bar{D}$ be any $\gamma_t^{oi}(\bar{G})$-set. The vertices of $V(G) \setminus D$ form a complete subgraph of $\bar{G}$, thus at most one of them does not belong to $\bar{D}$ as $V(\bar{G}) \setminus \bar{D}$ is independent. Therefore $|\bar{D}| \geq |V(G) \setminus D| - 1$. We now get $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) = |D| + \gamma_t^{oi}(\bar{G}) \geq |D| + |V(G) \setminus D| - 1 = n - 1$. Obviously, $\gamma_t^{oi}(G) \leq n$ and $\gamma_t^{oi}(\bar{G}) \leq n$. If $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) = 2n$, then $\gamma_t^{oi}(G) = n$ and $\gamma_t^{oi}(\bar{G}) = n$. By Theorem 18 we have $G = m_1K_2$ and $\bar{G} = m_2K_2$. Since this is not possible, we have $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) < 2n$. ■

Now let us observe that the bounds from the previous theorem are tight. For the lower bound, consider the graph $G_{1,1,1}$ of the family $\mathcal{G}$ and its complement $\overline{G_{1,1,1}}$, both given in Figure 2. Let us observe that the graphs $G_{1,1,1}$ and $\overline{G_{1,1,1}}$ are isomorphic. We have $\gamma_t^{oi}(G_{1,1,1}) + \gamma_t^{oi}(\overline{G_{1,1,1}}) = 2\gamma_t^{oi}(G_{1,1,1}) = 4 = n - 1$. For the upper bound, consider the cycle C_4 and its complement $\overline{C_4} = P_2 \cup P_2$. We have $\gamma_t^{oi}(C_4) + \gamma_t^{oi}(\overline{C_4}) = \gamma_t^{oi}(C_4) + \gamma_t^{oi}(P_2 \cup P_2) = 3 + 4 = 2n - 1$.

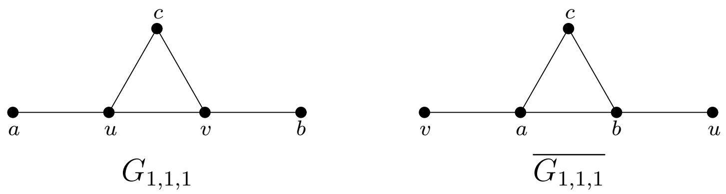

Figure 2: The graphs $G_{1,1,1}$ and $\overline{G_{1,1,1}}$

We now prove that the cycle on four vertices and its complement are the only graphs which attain the upper bound from Theorem 32.

Theorem 33 *Let G be a graph. We have $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) = 2n - 1$ if and only if $G = C_4$ or $\bar{G} = C_4$.*

Proof. Obviously, $\gamma_t^{oi}(C_4) + \gamma_t^{oi}(\bar{C}_4) = 7 = 2n - 1$. Now assume that for some graph G we have $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) = 2n - 1$. Without loss of generality we assume that $\gamma_t^{oi}(G) = n$. By Theorem 18 we have $G = mK_2$. Clearly, $m \geq 2$ as otherwise $\bar{G}$ has an isolated vertex. If $m = 2$, then $G = K_2 \cup K_2$, and consequently, $\bar{G} = C_4$. Now assume that $m \geq 3$. Let $E(mK_2) = \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$. Observe that $\{u_2, v_2, u_3, v_3, \ldots, u_m, v_m\}$ is a TOIDS of the graph $\overline{mK_2}$. Therefore $\gamma_t^{oi}(\overline{mK_2}) \leq n - 2$. We now get $\gamma_t^{oi}(mK_2) + \gamma_t^{oi}(\overline{mK_2}) \leq n + n - 2 < 2n - 1$, a contradiction. ■

Corollary 34 *If graphs G and $\bar{G}$ are different from C_4, then $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G})$ $\leq 2n - 2$.*

We now characterize all graphs, which attain the bound from the previous corollary.

Theorem 35 *Let G be a graph. We have $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) = 2n - 2$ if and only if $G = mK_2$ or $\bar{G} = mK_2$, where $m \geq 3$.*

Proof. If $G = mK_2$, then $\gamma_t^{oi}(G) = n$. It is not very difficult to verify that $\gamma_t^{oi}(\bar{G}) = n - 2$. Now assume that for some graph G we have $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) = 2n - 2$. Without loss of generality we may assume that $\gamma_t^{oi}(G) \geq \gamma_t^{oi}(\bar{G})$. Thus either $\gamma_t^{oi}(G) = n - 1$ and $\gamma_t^{oi}(\bar{G}) = n - 1$, or $\gamma_t^{oi}(G) = n$ and $\gamma_t^{oi}(\bar{G}) = n - 2$. First assume that $\gamma_t^{oi}(G) = n - 1$. Obviously, some of the graphs G or $\bar{G}$ is connected. Without loss of generality assume that G is connected. Theorem 20 implies that $G \in \{P_3, C_4, C_5\}$ or G is a complete graph on at least three vertices. It is not difficult to verify that $\gamma_t^{oi}(\bar{G}) \neq n - 1$. Now assume that $\gamma_t^{oi}(G) = n$ and $\gamma_t^{oi}(\bar{G}) = n - 2$. By Theorem 18 we have $G = mK_2$. It is not very difficult to verify that $\gamma_t^{oi}(\bar{G}) = n - 2$ only for $m \geq 3$. ■

Corollary 36 *If $G \neq mK_2$ and $\bar{G} \neq mK_2$, for any $m \geq 2$, then $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) \leq 2n - 3$.*

We now improve the lower bound from Theorem 32.

Theorem 37 *For every graph G with l leaves we have $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) \geq n + l - 2$.*

Proof. Let D and $\bar{D}$ be any $\gamma_t^{oi}(G)$-set and $\gamma_t^{oi}(\bar{G})$-set, respectively. If some leaf of G does not belong to the set $\bar{D}$, then $\gamma_t^{oi}(\bar{G}) \geq n - 2$, and we easily obtain the result. Now assume that all leaves of G belong to the set $\bar{D}$. Since $V(G) \setminus D$ is independent, the vertices of $V(G) \setminus D$ form a clique in $\bar{G}$. Consequently, at most one of them does not belong to $\bar{D}$ as $V(\bar{G}) \setminus \bar{D}$ is independent. Therefore $|\bar{D}| \geq |V(G) \setminus D| + l - 1$. We now get $\gamma_t^{oi}(G) + \gamma_t^{oi}(\bar{G}) \geq |D| + |V(G) \setminus D| + l - 1 > n + l - 2$. ■

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