Domination stability in graphs

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Abstract

For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G is the minimum cardinality of a dominating set of G. The domination stability, or just γ-stability, of a graph G is the minimum number of vertices whose removal changes the domination number. We show that the γ-stability problem is NP-complete, even when restricted to bipartite graphs. We obtain several bounds, exact values and characterizations for the γ-stability of a graph, and we characterize the trees with $st_\gamma(T) = 2$.
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1 Introduction

Let $G = (V, E)$ be a graph. The open neighborhood of a vertex v of G is the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. The private neighborhood $pn_G(v, S)$ of a vertex $v \in S$ is defined by $pn_G(v, S) = \{u \in V(G)\colon N_G(u) \cap S = \{v\}\}$. Each vertex in $pn_G(v, S)$ is called a private neighbor of v. The degree of a vertex v, that is, the cardinality of its open neighborhood, is denoted by $d_G(v)$. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The maximum (minimum, respectively) degree among all vertices of G we denote by $\Delta(G)$ ($\delta(G)$,

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respectively). The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. The complete graph on n vertices we denote by K_n. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Double star is a graph obtained from a star by joining a positive number of vertices to one of the leaves. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. By contracting the edge uv we mean replacing uv and the vertices u and v with a new vertex adjacent to all neighbors of u or v in G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by the vertices of S.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of minimum cardinality is called a $\gamma(G)$-set. For a comprehensive survey of domination in graphs, see [7].

A domination-critical (domination-super critical, respectively) vertex in a graph G is a vertex whose removal decreases (increases, respectively) the domination number. One of important problems in domination theory is to determine graphs in which every vertex is critical, see for example [1, 2, 6, 9, 10]. Much have been also written about graphs with no critical vertex, and for references, see for example [3, 4, 8].

Bauer et al. [1] introduced the concept of domination stability in graphs. The domination stability, or just γ-stability, of a graph G is the minimum number of vertices whose removal changes the domination number. The γ^--stability of G, denoted by $\gamma^-(G)$, is defined as the minimum number of vertices whose removal decreases the domination number, and the γ^+-stability of G, denoted by $\gamma^+(G)$, is defined as the minimum number of vertices whose removal increases the domination number. We denote the γ-stability of G by $st_\gamma(G)$. Thus the domination stability of a graph G is $st_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$.

We show that the γ-stability problem is NP-complete, even when restricted to bipartite graphs. We obtain several bounds, exact values and characterizations for the γ-stability of a graph, and we characterize the trees with $st_\gamma(T) = 2$.

It can be easily seen that if G is a disconnected graph with components $G_1, \dots, G_k$, then $st_\gamma(G) = \min\{st_\gamma(G_1), \dots, st_\gamma(G_k)\}$. Hence we only study connected graphs.

For a graph G, let $\rho(G) = \min\{|pn_G(v, S)| \colon v \in S,\ S \text{ is a } \gamma(G)\text{-set}\}$.

Bauer et al. obtained the following necessary and sufficient condition for a graph to have a domination-critical vertex.

Proposition 1 ([1]) *A graph G has a domination-critical vertex if and only if $\rho(G) = 0$.*

The following upper bound is known for the γ-stability of any graph.

Proposition 2 ([1]) *For every graph G we have $st_\gamma(G) \leq \delta(G) + 1$.*

2 Complexity

This section concerns the NP-completeness of the γ-stability decision problem.

DOMINATION STABILITY PROBLEM
INSTANCE: A graph $G = (V, E)$ and the domination number $\gamma(G)$.
QUESTION: Is $st_\gamma(G) > 1$?

Dettlaff et al. [5] studied the complexity of domination subdivision numbers of graphs. The domination subdivision number $sd(G)$ of a graph G is the minimum number of edges of G that must be subdivided (where an edge can be subdivided only once) in order to increase the domination number. Dettlaff et al. proved that the decision problem for domination subdivision number is NP-complete even for bipartite graphs (see Theorem 1 of [5]). Their proof was performed by a transformation from 3-SAT and usage of a gadget. With a similar proof using the same gadget and a transformation from 3-SAT, we can obtain the following result.

Theorem 3 *Domination stability problem is NP-complete even for bipartite graphs.*

Since the class of graphs with $st_\gamma(G) > 1$ is a subclass of graphs with no critical vertex, we have the following result.

Theorem 4 *The decision problem for determining graphs with no critical vertex is NP-complete even for bipartite graphs.*

3 Exact values

In this section we determine the domination stability for some classes of graphs.

Observation 5 *If G is a star or a double star, then $st_\gamma(G) = 1$.*

Observation 6 *For complete bipartite graphs $K_{m,n}$ with $2 \leq m \leq n$ we have $st_\gamma(K_{m,n}) = m - 1$.*

Observation 7 *We have* $\gamma(P_n) = \gamma(C_n) = \lfloor (n+2)/3 \rfloor$*.*

First we investigate the γ-stability of paths.

Proposition 8 *For paths* P_n *we have* $st_\gamma(P_n) = 2$ *if* $n \equiv 2$ *(mod 3), and* $st_\gamma(P_n) = 1$ *otherwise.*

Proof. First assume that $n \equiv 0$ (mod 3). Let us observe that $\gamma(P_n - v) = \gamma(P_n)$ $+1$, where v is a support vertex. Consequently, $st_\gamma(P_n) = 1$. Next assume that $n \equiv 1$ (mod 3). Then $\gamma(P_n - v) = \gamma(P_n) - 1$, where v is a leaf. Consequently, $st_\gamma(P_n) = 1$. Now assume that $n = 3k + 2$ for some integer k. By Observation 7 we have $\gamma(P_n) = k + 1$. Let v be an arbitrary vertex of P_n. We show that the removal of v does not change the domination number. If v is a leaf, then $\gamma(P_n - v) = \gamma(P_{n-1}) = k + 1 = \gamma(P_n)$. Now assume that the degree of v is 2. Let P_{n_1} and P_{n_2} be the components of $P_n - v$. Without loss of generality we may consider the following two cases. First assume that $n_1 \equiv 0$ (mod 3) and $n_2 \equiv 1$ (mod 3). Then we get $\gamma(P_n - v) = \gamma(P_{n_1}) + \gamma(P_{n_2}) = \lfloor (n_1 + 2)/3 \rfloor$ $+\lfloor (n_2 + 2)/3 \rfloor = n_1/3 + (n_2 + 2)/3 = (n + 1)/3 = k + 1 = \gamma(P_n)$. Now assume that $n_1 \equiv 2$ (mod 3) and $n_2 \equiv 2$ (mod 3). Then similarly we obtain $\gamma(P_n - v)$ $= \gamma(P_{n_1}) + \gamma(P_{n_2}) = \gamma(P_n)$. We conclude that $st_\gamma(P_n) \geq 2$. Now, Proposition 2 implies that $st_\gamma(P_n) = 2$. ■

Next we investigate the γ-stability of cycles.

Proposition 9 *We have* $st_\gamma(C_n) = i$ *if* $n \equiv i$ *(mod 3) for* $i = 1, 2$*, while* $st_\gamma(C_n) = 3$ *if* $n \equiv 0$ *(mod 3).*

Proof. First assume that $n = 3k + 1$ for some integer k. Then for any vertex v we have $\gamma(C_n - v) = \gamma(P_{n-1}) = k = \gamma(C_n) - 1$, and thus $st_\gamma(C_n) = 1$. Now assume that $n = 3k + 2$ for some integer k. For any vertex v we have $\gamma(C_n - v)$ $= \gamma(P_{n-1}) = k+1 = \gamma(C_n)$. Thus $st_\gamma(C_n) \geq 2$. Now $\gamma(C_n - u - v) = \gamma(P_{n-2}) = k$ $= \gamma(C_n) - 1$, where u and v are two adjacent vertices. This implies that $st_\gamma(C_n)$ $= 2$. Finally assume that $n = 3k$. It is easy to observe that the removal of any vertex does not change the domination number. Since $C_n - v = P_{n-1}$ and by Proposition 8 we have $st_\gamma(P_{n-1}) = 2$, we conclude that $st_\gamma(C_n) = 3$. ■

4 Bounds

In this section we present several sharp bounds and characterizations for the domination stability of a graph.

Proposition 10 *If* G *is a graph of order* n*, then* $st_\gamma(G) \leq n$ *with equality if and only if* $G = K_n$*.*

Proof. The bound for an arbitrary graph is obvious. Clearly, $st_\gamma(K_n) = n$. Now suppose that $G \neq K_n$ is a graph of order n with $st_\gamma(G) = n$. Obviously, $n \geq 2$. First assume that $\gamma(G) = 1$. Let u and v be a pair of non-adjacent vertices. Since $\gamma(G) = 1$, we have $N_G(u) \cap N_G(v) \neq \emptyset$. The graph obtained from G by removing all common neighbors of u and v has domination number larger than 1, contradicting $st_\gamma(G) = n$. Now assume that $\gamma(G) \geq 2$. Let D be a $\gamma(G)$-set. We can assume that every vertex of D has a private neighbor in $V(G) \setminus D$, since $st_\gamma(G) = n$. Let $x \in D$. Then the graph obtained by removing x and its private neighbors in $V(G) \setminus D$ has domination number less than $\gamma(G)$, contradicting $st_\gamma(G) = n$. ■

Theorem 11 *For any graph G with $\gamma(G) \geq 2$ we have $st_\gamma(G) \leq \lfloor n/\gamma(G) \rfloor$, and this bound is sharp.*

Proof. Let D be a $\gamma(G)$-set, and let x be a vertex of D with minimum number of private neighbors in $V(G) \setminus D$. Then the removal of x and its private neighbors in $V(G) \setminus D$ implies that $st_\gamma(G) \leq \lfloor n/\gamma(G) \rfloor$. To see the sharpness, consider a cycle C_n, where $n \equiv 2 \pmod 3$. ■

Theorem 12 ([7]) *If G is a connected graph of order n with no isolated vertex, then $\gamma(G) \leq n/2$ with equality if and only if G is the cycle C_4 or the corona HoK_1 for some connected graph H.*

It can be easily seen that $st_\gamma(G) = 1$ if $\gamma(G) = n/2$. Let G be a connected graph with $\gamma(G) \geq 2$ and $1 \leq k < 2\gamma(G) - 2$. By Theorem 12 we have $n > kn/2(\gamma(G) - 1) \geq k\gamma(G)/(\gamma(G) - 1)$. Then we obtain the following result.

Corollary 13 *If $\gamma \geq 2$ and $1 \leq k < 2\gamma - 2$, then there is no connected graph G of order n with $\gamma(G) = \gamma$ and $st_\gamma(G) = n - k$.*

We now prove that the domination stability of no graph is one less than its order.

Proposition 14 *There is no connected graph G of order n with $st_\gamma(G) = n - 1$.*

Proof. Suppose that G is a graph of order n with $st_\gamma(G) = n - 1$. Corollary 13 implies that $\gamma(G) = 1$. Let x be a universal vertex of G. From $st_\gamma(G) = n - 1$ we obtain that there is no pair of non-adjacent vertices in $N_G(x)$. Consequently, $G = K_n$. But then $st_\gamma(G) = n$, a contradiction. ■

Proposition 15 *For every integers n and k such that $1 \leq k \leq n$ and $k \neq n - 1$ there exists a graph G of order n with $st_\gamma(G) = k$.*

Proof. We construct a graph $G_{k,n}$ from a complete graph K_k with vertices $v_1, v_2, \ldots, v_k$ by adding $n-k$ new vertices $u_1, u_2, \ldots, u_{n-k}$ together with new edges v_iu_j, $1 \le i \le k$, $1 \le j \le n-k$. It can be easily verified that the removal of any subset of $k-1$ vertices of $G_{k,n}$ does not change the domination number. On the other hand, $\gamma(G_{k,n} - v_1 - v_2 - \ldots - v_k) = n-k \neq 1$. Thus $st_\gamma(G_{k,n}) = k$. ■

We have the following characterization of graphs with domination stability two less than the order.

Proposition 16 *For a connected graph G of order n we have $st_\gamma(G) = n-2$ if and only if $G = G_{n-2,n}$.*

Proof. It was already seen that $st_\gamma(G_{n-2,n}) = n-2$. Now let G be a graph of order n with $st_\gamma(G) = n-2$. Corollary 13 implies that $\gamma(G) = 1$. Clearly, $G \neq K_n$. Let y and z be two non-adjacent vertices of G. If there is a vertex $x \in V(G) \setminus \{y,z\}$ such that $x \notin N_G(y) \cap N_G(z)$, then the removal of x, y and z yields that $st_\gamma(G) < n-2$, a contradiction. Thus every vertex of $V(G) \setminus \{y,z\}$ is adjacent to both y and z. If there are two non-adjacent vertices x_1 and x_2 in $V(G) \setminus \{y,z\}$, then similarly we obtain $st_\gamma(G) < n-2$, a contradiction. Thus the graph induced by $V(G) \setminus \{y,z\}$ is complete. Consequently, $G = G_{n-2,n}$. ■

Theorem 17 *There is no forbidden induced subgraph characterization for a connected graph G of order n with $st_\gamma(G) = k$, where $1 \le k \le n/\gamma(G)$.*

Proof. Let H be a graph of order m. For any vertex $v \in V(H)$, we identify v with a vertex of a complete graph K_{k+1}, to obtain a graph G. Note that G has order $m(k+1)$. One can verify that $\gamma(G) = n$ and $st_\gamma(G) = k$. ■

We have the following upper bound on the domination stability of a graph with domination number at least two.

Proposition 18 *For any graph G with $\gamma(G) \ge 2$ we have $st_\gamma(G) \le \min\{\delta(G) +1, n - \delta(G) - 1\}$.*

Proof. Let D be a $\gamma(G)$-set. If there is a vertex $x \in D$ such that $pn(x,D) = \emptyset$, then x is a domination-critical vertex and $st_\gamma(G) = 1$. Thus assume that for every vertex $x \in D$ we have $pn(x,D) \neq \emptyset$. Let $v \in D$, and let A be the set of all private neighbors of v in $V(G) \setminus D$. Then $\gamma(G[A \cup \{v\}]) = 1 < \gamma(G)$. Using Proposition 2 we get $st_\gamma(G) \le \min\{\delta(G) + 1, n - \delta(G) - 1\}$. ■

We next give Nordhaus-Gaddum type inequalities for the sum of the domination stabilities of a graph and its complement. First note that if $\gamma(G) = 1$, then $st_\gamma(\overline{G}) = 1$, and thus we obtain the following bound, which is sharp for complete graphs.

Observation 19 *If* G *is a graph with* $\gamma(G) = 1$ *or* $\gamma(\overline{G}) = 1$*, then* $st_\gamma(G) + st_\gamma(\overline{G}) \leq n+1$*, and this bound is sharp.*

Theorem 20 *If* G *is a graph with* $\gamma(G) \geq 2$ *and* $\gamma(\overline{G}) \geq 2$*, then* $st_\gamma(G) + st_\gamma(\overline{G}) \leq n-1$*, and this bound is sharp.*

Proof. It is obvious that if $\delta(G) \leq (n-2)/2$, then $\min\{\delta(G)+1, n-\delta(G)-1\} \leq n/2$ and $\min\{\delta(\overline{G})+1, n-\delta(\overline{G})-1\} \leq (n-2)/2$. If $\delta(G) > (n-2)/2$, then $\min\{\delta(G)+1, n-\delta(G)-1\} \leq (n-2)/2$ and $\min\{\delta(\overline{G}), n-\delta(\overline{G})-1\} \leq n/2$. By Proposition 18 we have $2 \leq st_\gamma(G) + st_\gamma(\overline{G}) \leq n-1$. To see the sharpness, consider a cycle C_4. ■

5 Trees

It follows from Proposition 2 that $st_\gamma(T) \in \{1,2\}$, for any tree T. In this section we present a constructive characterization of trees T with $st_\gamma(T) = 2$. For this purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_2. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a path P_2 by joining one of its vertices to a vertex of T_k adjacent to at least two support vertices of degree two.
- Operation $\mathcal{O}_2$: Attach a path P_5 by joining the vertex of minimum eccentricity to any vertex of T_k.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaves to a vertex of T_k, which is a leaf or a support vertex or is adjacent to a path P_3.

We now prove that the γ-stability of every tree of the family $\mathcal{T}$ equals two.

Lemma 21 *If* $T \in \mathcal{T}$*, then* $st_\gamma(T) = 2$.

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_2$, then obviously $st_\gamma(T) = 2$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. The vertex to which is attached P_2 we denote by x. Let v_1v_2 be the attached path. Let v_1 be joined to x. Let ab and cd denote paths P_2 adjacent to x and different from v_1v_2. Let x be adjacent to a and c. Let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v_1\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')+1$. Now let D be a $\gamma(T)$-set that contains all support vertices. We have $v_1, a \in D$. It is easy to observe that $D \setminus \{v_1\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. We now show that $st_\gamma(T) = 2$. On the contrary, suppose that $st_\gamma(T) = 1$. Let v be a vertex of T such that $\gamma(T - v) \neq \gamma(T)$. By the similarity of the paths $v_1 v_2$ and ab, without loss of generality we may assume that $v \neq v_1, v_2$. Let D' be a $\gamma(T' - v)$-set. It is easy to see that $D' \cup \{v_1\}$ is a DS of the graph $T - v$. Thus $\gamma(T - v) \leq \gamma(T' - v) + 1$. Now let D be a $\gamma(T - v)$-set that contains all support vertices. Let us observe that $D \setminus \{v_1\}$ is a DS of the graph $T' - v$ as the vertex x is still dominated. Therefore $\gamma(T' - v) \leq \gamma(T - v) - 1$. This implies that $\gamma(T - v) = \gamma(T' - v) + 1$. We now get $\gamma(T' - v) = \gamma(T - v) - 1 \neq \gamma(T) - 1 = \gamma(T')$. Consequently, $st_\gamma(T') = 1$. This is a contradiction, and hence $st_\gamma(T) = 2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which is attached P_5 we denote by x. Let $v_1 v_2 v_3 v_4 v_5$ be the attached path. Thus v_3 is joined to x. Let D' be a $\gamma(T')$-set. It is easy to observe that $D' \cup \{v_2, v_4\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. Now let us observe that there exists a $\gamma(T)$-set that contains every support vertex, and does not contain the vertex v_3. Let D be such a set. Observe that $D \setminus \{v_2, v_4\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 2$. This implies that $\gamma(T) = \gamma(T') + 2$. Suppose that $st_\gamma(T) = 1$. Let v be a vertex of T such that $\gamma(T - v) \neq \gamma(T)$. Let us observe that $v \notin \{v_1, v_2, v_3, v_4, v_5\}$. Thus $v \in V(T')$. It is easy to observe that $\gamma(T - v) = \gamma(T' - v) + 2$. We now get $\gamma(T' - v) = \gamma(T - v) - 2 \neq \gamma(T) - 2 = \gamma(T')$. This implies that $st_\gamma(T') = 1$, a contradiction. Consequently, $st_\gamma(T) = 2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. The vertex to which is attached P_3 we denote by x. Let $v_1 v_2 v_3$ be the attached path. We first prove that there exists a $\gamma(T')$-set that contains the vertex x. If x is a support vertex, then the claim is obvious. Now assume that x is adjacent to a path P_3, say abc. Let a and x be adjacent. Since $st_\gamma(T') = 1$, we have $\gamma(T' - b) = \gamma(T')$. Let us observe that every $\gamma(T' - b)$-set D is a $\gamma(T')$-set as the vertex b is dominated by the vertex c. Since x is a support vertex of the graph $T' - b$, we may assume that $x \in D$. Now assume that x is a leaf of the tree T'. The neighbor of x we denote by y. Since $st_\gamma(T') = 1$, we have $\gamma(T' - y) = \gamma(T')$. Let D be a $\gamma(T' - y)$-set. Clearly, $D \cup \{y\} \setminus \{x\}$ is a $\gamma(T')$-set. We now conclude that there exists a $\gamma(T')$-set that contains the vertex x. Now let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v_2\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma(T)$-set that contains all support vertices, and does not contain the vertex v_1. Observe that $D \setminus \{v_2\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. Suppose that $st_\gamma(T) = 1$. Let v be a vertex of T such that $\gamma(T - v) \neq \gamma(T)$. Clearly, $\gamma(T - v_1) = \gamma(T') + \gamma(P_2) = \gamma(T') + 1 = \gamma(T)$. Thus $v \neq v_1$. Let D be a $\gamma(T - v)$-set. If $v = v_2$, then $v_3 \in D$. It is easy to see that D is a DS of the tree T, and consequently, $\gamma(T) \leq \gamma(T - v)$. Now let D' be a $\gamma(T')$-set that contains the vertex x. Let us observe that $D' \cup \{v_3\}$ is a DS of the graph $T - v_2$. Therefore $\gamma(T - v) \leq \gamma(T') + 1 = \gamma(T)$, and consequently, $\gamma(T - v) = \gamma(T)$, which is a contradiction. Thus $v \neq v_2$. Now assume that

$v = v_3$. Clearly, $\gamma(T - v) \leq \gamma(T)$. If $\gamma(T - v) < \gamma(T)$, then using Proposition 1, there is a $\gamma(T)$-set D such that $v_3 \in D$ and $pn_T(v_3, D) = \emptyset$. We may assume that $v_2 \notin D$ and $v_1 \in D$. If $x \in D$, then $D \cup \{v_2\} \setminus \{v_1, v_3\}$ is a DS of the tree T of cardinality less than $|D|$, which is a contradiction. Thus $x \notin D$. Now, $D \cup \{x\} \setminus \{v_1, v_3\}$ is a $\gamma(T')$-set containing x, and x has no private neighbor in $V(T') \setminus D$, a contradiction to Proposition 1. Thus we now assume that $v \in V(T')$. Clearly, $\gamma(T - v) = \gamma(T' - v) + 1$. Hence $st_\gamma(T') = 1$, a contradiction. Therefore $st_\gamma(T) = 2$. ■

We now prove that if the γ-stability of a tree equals two, then the tree belongs to the family $\mathcal{T}$.

Lemma 22 *Let T be a tree. If $st_\gamma(T) = 2$, then $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 0$, then $T = P_1$. We have $st_\gamma(P_1) = 1$. If $\mathrm{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. Now assume that $\mathrm{diam}(T) \in \{2, 3\}$. Thus T is a star or a double star. It is not difficult to observe that $st_\gamma(T) = 1$. Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order n of the tree T is at least five. We obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z be leaves adjacent to x. Let $T' = T - x$. Let D' be a $\gamma(T')$-set. We have $y, z \in D'$. It is easy to observe that $D' \setminus \{y, z\} \cup \{x\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') - 1$, and consequently, $\gamma(T') > \gamma(T)$. This implies that $st_\gamma(T) = 1$, a contradiction. Thus every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that $d_T(u) \geq 3$. Let y be a child of u other than v. First assume that y is a leaf. Let $T' = T - t$. Let D be a $\gamma(T)$-set that contains all support vertices. Let us observe that $D \setminus \{v\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. This implies that $st_\gamma(T) = 1$, a contradiction.

Thus every child of u is a support vertex of degree two. The leaf adjacent to y we denote by z. First assume that $d_T(u) \geq 4$. Let $T' = T - T_v$. Let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma(T)$-set that contains all support vertices. Let us observe that $D \setminus \{v\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. We now show that $st_\gamma(T') = 2$. On the contrary, suppose that $st_\gamma(T') = 1$. Let x be a vertex of T' such that $\gamma(T' - x) \neq \gamma(T')$. Clearly, $\gamma(T - x) \leq \gamma(T' - x) + 1$. Now let D be a $\gamma(T - x)$-set that contains all support vertices. Let us observe that $D \setminus \{v\}$ is a DS of the graph $T' - x$. Therefore $\gamma(T' - x) \leq \gamma(T - x) - 1$. Consequently, $\gamma(T - x) = \gamma(T' - x) + 1$.

We now get $\gamma(T - x) = \gamma(T' - x) + 1 \neq \gamma(T') + 1 = \gamma(T)$, a contradiction to $st_\gamma(T) = 2$. Therefore $st_\gamma(T') = 2$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 3$. Let $T' = T - T_u$. Let D' be a $\gamma(T')$-set. It is easy to observe that $D' \cup \{v, y\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')+2$. Now let D be a $\gamma(T)$-set that contains all support vertices, and does not contain the vertex u. Observe that $D \setminus \{v, y\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 2$. This implies that $\gamma(T) = \gamma(T') + 2$. We now show that $st_\gamma(T') = 2$. Suppose that $st_\gamma(T') = 1$. Let x be a vertex of T' such that $\gamma(T' - x) \neq \gamma(T')$. Let us observe that $\gamma(T - x) = \gamma(T' - x) + 2$. We now get $\gamma(T - x) = \gamma(T' - x) + 2 \neq \gamma(T') + 2$ $= \gamma(T)$, a contradiction to $st_\gamma(T) = 2$. Therefore $st_\gamma(T') = 2$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. First assume that there is a child of w, say k, such that the distance of w to the most distant vertex of T_k is two. It suffices to consider only the possibility when T_k is a path P_2, say kl. Assume that there exists a $\gamma(T)$-set that contains the vertex w. Let D be such a set. We can assume that $v, k \in D$. Let $T' = T - l$. Let us observe that $D \setminus \{k\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. Consequently, $\gamma(T') \neq \gamma(T)$. Now assume that no $\gamma(T)$-set contains the vertex w. Let $T'' = T - v$. Let D'' be a $\gamma(T'')$-set that contains all support vertices. Clearly, $t \in D''$ as t is an isolated vertex. Let us observe that $D'' \cup \{v\} \setminus \{t\}$ is a DS of the tree T. Since no $\gamma(T)$-set contains the vertex w, the set $D'' \cup \{v\} \setminus \{t\}$ is not minimum. Therefore $\gamma(T) \leq \gamma(T'') - 1$. Consequently, $\gamma(T'') \neq \gamma(T)$. We now conclude that $st_\gamma(T) = 1$, a contradiction.

Now assume that there is no child of w, say k, such that the distance of w to the most distant vertex of T_k is two. Let $T' = T - T_u$. Let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Now let D be a $\gamma(T)$-set that contains all support vertices, and does not contain the vertex u. Observe that $D \setminus \{v\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. We now show that $st_\gamma(T') = 2$. Suppose that $st_\gamma(T') = 1$. Let x be a vertex of T' such that $\gamma(T' - x) \neq \gamma(T')$. Clearly, $\gamma(T - x) = \gamma(T' - x) + 1$. We now get $\gamma(T - x) = \gamma(T' - x) + 1 \neq \gamma(T') + 1 = \gamma(T)$, a contradiction to $st_\gamma(T) = 2$. Therefore $st_\gamma(T') = 2$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$. ∎

As an immediate consequence of Lemmas 21 and 22, we have the following characterization of trees with γ-stability equaling two.

Theorem 23 *Let T be a tree. Then $st_\gamma(T) = 2$ if and only if $T \in \mathcal{T}$.*

6 Open problems

Theorem 4 states that the decision problem for determining graphs with no critical vertex is NP-complete even for bipartite graphs. A graph is domination dot-critical if contracting any edge decreases the domination number. A graph is domination bicritical if the removal of any pair of vertices decreases the domination number. There are still several open problems on determining graphs with no critical vertex, which are domination dot-critical or domination bicritical, see [3, 4].

Problem 24 *Determine the complexity issue of domination dot-critical graphs (domination bicritical graphs) with no critical vertex.*

Problem 25 *Characterize all graphs achieving the bound of Theorem 11.*

References

[1] D. Bauer, F. Harary, J. Nieminen and C. Suffel, *Domination alternation sets in graphs,* Discrete Mathematics 47 (1983), 153–161.

[2] R. Brigham, P. Chinn and R. Dutton, *Vertex domination-critical graphs,* Networks 18 (1988), 173–179.

[3] R. Brigham, T. Haynes, M. Henning and D. Rall, *Bicritical domination,* Discrete Mathematics 305 (2005), 18–32.

[4] T. Burton and D. Sumner, *Domination dot-critical graphs,* Discrete Mathematics 306 (2006), 11–18.

[5] M. Dettlaff, J. Raczek and J. Topp, *Domination subdivision and domination multisubdivision numbers of graphs,* submitted.

[6] O. Favaron, D. Sumner and E. Wojcicka, *The diameter of domination-critical graphs,* Journal of Graph Theory 18 (1994), 723–734.

[7] T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

[8] M. Krzywkowski and D. Mojdeh, *Bicritical domination and duality coalescence of graphs,* submitted.

[9] D. Sumner and P. Blitch, *Domination critical graphs,* Journal of Combinatorial Theory Series B 34 (1983), 65–76.

[10] D. Sumner, *Critical concept in domination,* Discrete Mathematics 86 (1990), 33–46.