On the intersection graphs of subspaces of a vector space

Nader Jafari Rad* n.jafarirad@gmail.com
Sayyed Heidar Jafari* shjafari55@gmail.com

Marcin Krzywkowski†‡ marcin.krzywkowski@gmail.com

Abstract

We consider the intersection graph of subspaces of a vector space. We characterize all vector spaces whose intersection graph is connected, bipartite, complete, Eulerian, or planar.
Keywords: vector space, subspace, dimension, intersection graph, connected, complete, bipartite, Eulerian, planar.
AMS **Subject Classification:** 15A03, 05C75, 05C62, 05C69.

1 Introduction

For graph theory terminology in general we follow [9]. Specifically, let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ of order n and edge set $E(G)$. We denote the degree of a vertex v in G by $d_G(v)$, which is the number of edges incident to v. A graph G is *complete* if there is an edge between every pair of the vertices of G. A subset X of the vertices of a graph G is called *independent* if there is no edge with two endpoints in X. A graph G is called *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge of G has one endpoint in X and the other endpoint in Y. A graph G is called *complete*

*Department of Mathematics, Shahrood University of Technology, Iran
†Research fellow of the Claude Leon Foundation at the Department of Pure and Applied Mathematics, University of Johannesburg, South Africa.
‡Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Poland. Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

if any two different vertices of G are adjacent. A complete graph on n vertices is denoted by K_n. The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv\colon uv \notin E(G)\}$. The *null* graph on n vertices is $\overline{K_n}$. A *path* of length n is an ordered list of distinct vertices $v_0, v_1, \ldots, v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. We refer $v_0 - v_1 - \ldots - v_n$ as a such path. A (u, v)-path is a path with endpoints u and v. A *cycle* is a path $v_0, v_1, \ldots, v_n$ with the extra edge v_0v_n. For vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$, and $d(x, y) = \infty$ if there is no path between x and y). The *diameter* $diam(G)$ of G is maximum number $d(x, y)$, over all $x, y \in V(G)$. A graph G is *connected* if it has a (u, v)-path for each pair $u, v \in V(G)$. A graph is *Eulerian* if it has a closed trail containing all edges. A graph is *planar* if it can be drawn in the plane such that its edges intersect only at their ends. A set of pairwise independent edges in a graph G is called a *matching*. A matching is *perfect* if it is incident with every vertex of G.

Let $F = \{S_i\colon i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ is the one-dimensional skeleton of the nerve of F, i.e., $G(F)$ is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$ [8].

Theorem 1 *(Marczewski [8]) Every simple graph is an intersection graph.*

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last few decades, leading to many fascinating results and questions. It is interesting to study the intersection graph $G(F)$ when the members of F have an algebraic structure. For the last few decades several mathematicians studied such graphs on various algebraic structures. These interdisciplinary studies allow us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa. The first step in this direction was taken by Bosak [2] in 1964. Then Cskny and Pollk [4] studied the graphs of subgroups of a finite group. Zelinka [10] continued the work on intersection graphs of nontrivial subgroups of finite abelian groups. Recently Chakrabarty et al. [3] studied intersection graphs of ideals of rings.

In this paper, we consider the intersection graph of subspaces of a vector space. For a vector space V, the *intersection graph of subspaces* of V, denoted by $G(V)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial subspaces of V and two distinct vertices are adjacent if and only if the corresponding subspaces of V have a nontrivial (nonzero) intersection. Clearly the set of vertices is empty if $dim(V) = 1$.

In section 2, we state known and preliminary results which we use in the next sections. In section 3, we determine vertex degrees in the intersection graphs of subspaces of a vector space. In section 4, we characterize all vector spaces V which $G(V)$ is Eulerian. In section 5, we characterize all vector spaces V which $G(V)$ is connected, bipartite, complete or planar.

Throughout this paper V is a vector space with $\dim(V) = n$ on a finite field F with $|F| = q$. We also denote by 0 the zero subspace of a vector space.

2 Known and preliminary results

In this section we state known and preliminary results which we use in the next sections.

Lemma 2 *Let V be a vector space and $W \leq V$. There is a subspace W' of V such that $V = W \bigoplus W'$.*

Proof. The proof follows from the fact that any linearly independent subset of V can extend to a basis of V. ■

Lemma 3 *Let V be a vector space with $\dim(V) < \infty$, and let $W \leq V$. Then the map $\phi\colon \{W' \colon W' \leq V, W' \cap W = 0\} \to \frac{V}{W}$ defined by $\phi(W') = \frac{W' \bigoplus W}{W}$ is surjective.*

Proof. Let T be a subspace of $\frac{V}{W}$. Then $T = \frac{W_1}{W}$, where $W \leq W_1 \leq V$. By Lemma 2 there is a subspace W' of V such that $W_1 = W \bigoplus W'$. So $T = \frac{W \bigoplus W'}{W}$. ■

Corollary 4 *If W, W' are two subspaces of V and $W \cap W' = 0$, then $\dim(W') =$ $dim(\frac{W' \bigoplus W}{W})$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_t\}$ be a basis for W'. It is easily seen that $\{a_1 + W, a_2 + W, \ldots, a_t + W\}$ is a basis for $\frac{W' \bigoplus W}{W}$. ■

Lemma 5 *([6]) For two subspaces U, W of V, $\dim(U+W) = \dim(U)+\dim(W)-\dim(U \cap W)$.*

Proposition 6 *Let W, W' be two subspaces of V with $\dim(W) = m$ and $\dim(W') = t$. Then $|\{W_1 \colon \dim(W_1) = t, W \bigoplus W' = W \bigoplus W_1\}| = q^{mt}$.*

Proof. Assume that $W' = \langle a_1, a_2, \ldots, a_t \rangle$. Let W_1 be an arbitrary subspace of V. We prove that $W_1 \in \{W_1 \colon \dim(W_1) = t, W \bigoplus W' = W \bigoplus W_1\}$ if and only if $W_1 = \langle b_1, b_2, \ldots, b_t \rangle$, where $b_i = a_i - w_i$, $w_i \in W$, and $i = 1, 2, \ldots, t$. Let $W_1 \in \{W_1 \colon \dim(W_1) = t, W \bigoplus W' = W \bigoplus W_1\}$. Then $W_1 + W = W' + W$. So for each $i \in \{1, 2, \ldots, t\}$, there are $b_i \in W_1, w_i \in W$ such that $a_i = b_i - w_i$. Let $U = \langle b_1, b_2, \ldots, b_t \rangle$. It is obvious that $U \subseteq W_1$ and $U + W = W_1 + W$. It follows from Lemma 5 that $U \cap W = 0$. So $U = W_1$. The converse is similarly verified. We now count $|\{W_1 \colon \dim(W_1) = t, W \bigoplus W' = \bigoplus W_1\}|$. For $i = 1, 2, \ldots, t$, there are q^m choices for w_i. Altogether, there are $(q^m)^t$ choices. ■

3 Vertex degrees

In this section we determine vertex degrees in the intersection graph of subspaces of a vector space. Let F be a finite field with $|F| = q$, and let V be an n-dimensional vector space over F. For integer $t \in \{1, 2, \ldots, n\}$, the number of t-dimensional subspaces of V is given in [5] by

$$\left[\begin{array}{c} n \\ t \end{array}\right]_q = \prod_{0 \leq i < t} \frac{q^{n-i} - 1}{q^{t-i} - 1}.$$

Assume that $\left[\begin{array}{c} n \\ 0 \end{array}\right]_q = 1$ and $\left[\begin{array}{c} n \\ t \end{array}\right]_q = 0$ if $t \notin \{0, 1, 2, \ldots, n\}$.

Lemma 7 *If* $\dim(W) = m$, *then*

$$|\{W' \colon \dim(W') = t, W \cap W' = 0\}| = q^{mt} \left[\begin{array}{c} n-m \\ t \end{array}\right]_q.$$

Proof. Follows from Lemma 3 and Proposition 6. ■

Corollary 8 *For* $\dim(W) = m$,

$$|\{W' \colon W \cap W' = 0\}| = \sum_{t=0}^{n-m} q^{mt} \left[\begin{array}{c} n-m \\ t \end{array}\right]_q.$$

A consequent of Corollary 8 leads to the following.

Theorem 9 *If* $\dim(W) = m$, *then*

$$\deg(W) = \sum_{t=0}^{n} \left[\begin{array}{c} n \\ t \end{array}\right]_q - \sum_{t=0}^{n-m} q^{mt} \left[\begin{array}{c} n-m \\ t \end{array}\right]_q - 2.$$

Corollary 10 *The degree of a vertex depends only to its dimension.*

Observation 11 *For two subspaces* W_1, W_2 *with* $\dim(W_1) \leq \dim(W_2)$, $\deg(W_1) \leq \deg(W_2)$.

Proof. Let $\dim(W_1) \leq \dim(W_2)$. There are two subspaces U_1, U_2 of V such that $\dim(U_1) = \dim(W_1)$, $\dim(U_2) = \dim(W_2)$, and $U_1 \subseteq U_2$. Since $deg(U_1) \leq \deg(U_2)$, Corollary 10 completes the proof. ■

Lemma 12 *For* $0 \leq t < n+1$, $\left[\begin{array}{c} n+1 \\ t \end{array}\right]_q = q^t \left[\begin{array}{c} n \\ t \end{array}\right]_q + \left[\begin{array}{c} n \\ t-1 \end{array}\right]_q$.

Corollary 13 $\delta(G(V)) = \sum_{t=0}^{n-2} \left[\begin{array}{c} n-1 \\ t \end{array}\right]_q - 1$, *and*

$$\Delta(G(V)) = \sum_{t=0}^{n} \left[\begin{array}{c} n \\ t \end{array}\right]_q - q^{n-1} - 3.$$

4 Eulerian graphs

In this section we characterize all vector spaces whose intersection graphs are Eulerian. It is well known that a graph G is Eulerian if and only of any vertex of G is of even degree.

Lemma 14 *Let $n_1, n_2, \ldots, n_t$ be positive odd numbers. For positives integers $m_1, m_2, \ldots, m_t$, $\sum_{i=1}^{t} m_i n_i$ is even if and only if $\sum_{i=1}^{t} m_i$ is even.*

Lemma 15 *For $n \geq 1$, q odd, $\sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q$ is even.*

Proof. If $n = 1$, then $\sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q = 2$. So we assume that $n \geq 2$. Notice that

$$\begin{aligned}
\sum_{t=0}^{n+1} \begin{bmatrix} n+1 \\ t \end{bmatrix}_q &= \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_q + \sum_{t=0}^{n} \begin{bmatrix} n+1 \\ t \end{bmatrix}_q \\
&= 1 + \sum_{t=0}^{n} (q^t \begin{bmatrix} n \\ t \end{bmatrix}_q + \begin{bmatrix} n \\ t-1 \end{bmatrix}_q) \\
&= 1 + \sum_{t=0}^{n} q^t \begin{bmatrix} n \\ t \end{bmatrix}_q + \sum_{t=0}^{n} \begin{bmatrix} n \\ t-1 \end{bmatrix}_q \\
&= \sum_{t=0}^{n} q^t \begin{bmatrix} n \\ t \end{bmatrix}_q + \sum_{t=0}^{n-1} \begin{bmatrix} n \\ t \end{bmatrix}_q + \begin{bmatrix} n \\ n \end{bmatrix}_q \\
&= \sum_{t=0}^{n} q^t \begin{bmatrix} n \\ t \end{bmatrix}_q + \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q.
\end{aligned}$$

Since $q^0, q^1, \ldots, q^n$ are odd, by Lemma 14, if $\sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q$ is even, then $\sum_{t=0}^{n} q^t \begin{bmatrix} n \\ t \end{bmatrix}_q$ is even, and so $\sum_{t=0}^{n+1} \begin{bmatrix} n+1 \\ t \end{bmatrix}_q$ is even. Now the result follows by an induction. ■

Theorem 16 *If q is even then $G(V)$ is Eulerian if and only if n is an even greater than 2.*

Proof. First notice that if $n = 2$, then $G(V)$ is not Eulerian. So we assume that $n \geq 3$. Let $W \leq V$, and $dim(W) = m$. Notice that

$$\begin{aligned}
\deg(W) &= \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q - \sum_{t=0}^{n-m} q^{mt} \begin{bmatrix} n-m \\ t \end{bmatrix}_q - 2 \\
&= \sum_{t=1}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q - \sum_{t=1}^{n-m} q^{mt} \begin{bmatrix} n-m \\ t \end{bmatrix}_q - 2
\end{aligned}$$

is even if and only if $\sum_{t=1}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q$ is even. But for any $t \in \{1, 2, \ldots, n\}$, $\begin{bmatrix} n \\ t \end{bmatrix}_q$ is odd. Now the result follows. ■

Theorem 17 *If q is odd, then $G(V)$ is Eulerian if and only if $n \geq 3$.*

Proof. First notice that if $n = 2$, then $G(V)$ is not connected and so is not Eulerian. Let $n \geq 3$. Let $W \leq V$ and $\dim(W) = m$, where $1 \leq m < n + 1$. By Theorem 9,

$$deg(W) = \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q - \sum_{t=0}^{n-m} q^{mt} \begin{bmatrix} n-m \\ t \end{bmatrix}_q - 2.$$

By Lemma 15, $\sum_{t=0}^{n-m} \begin{bmatrix} n-m \\ t \end{bmatrix}_q$ and $\sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q$ are even. But $q^0, q^m, \ldots, q^{m(n-m)}$ are odd. By Lemma 14, $\sum_{t=0}^{n-m} q^{mt} \begin{bmatrix} n-m \\ t \end{bmatrix}_q$ is even. We deduce that $\deg(W)$ is even. ■

5 Connected, bipartite and planar graphs

In this section we characterize all vector spaces whose intersection graphs are connected, bipartite, complete, or planar.

Lemma 18 *$G(V)$ is the null graph if and only if $\dim(V) = 2$.*

Proof. Let $G(V)$ be the null graph. If $\dim(V) \geq 3$, then V contains at least two linear independent vectors a_1, a_2. It is obvious that $\langle a_1, a_2 \rangle \neq V$. Now $\langle a_1, a_2 \rangle \cap \langle a_1 \rangle \neq 0$. This contradiction implies that $\dim(V) = 2$.

Conversely, suppose that $\dim(V) = 2$. Then any proper nontrivial subspace of V has dimension one. This implies that any proper nontrivial subspace of V is both minimal and maximal subspace. This means that $G(V)$ has no edges. ■

Theorem 19 *$G(V)$ is connected if and only if $\dim(V) \geq 3$.*

Proof. Let V be a vector space, and $\dim(V) = n$.

($\Longrightarrow$) Follows by Lemma 18.

($\Longleftarrow$) Let X, Y be two proper nontrivial subspaces of V. Let $a \in X$, $b \in Y$. Since $\dim(V) \geq 3$, $\langle a, b \rangle \neq V$. Then $\langle a, b \rangle \cap X \neq 0$ and $\langle a, b \rangle \cap Y \neq 0$. Now $X - \langle a, b \rangle - Y$ is a path between X and Y. We conclude that $G(V)$ is connected. ■

Lemma 20 *If* $\dim(V) \geq 3$, *then* $diam(G(V)) = 2$.

Proof. Since $\dim(V) \geq 3$, V contains two linear independent vectors a, b. Then $\langle a \rangle \cap \langle b \rangle = 0$. So $diam(G(V)) \geq 2$. On the other hand by the proof of Theorem 19, any two vertex of $G(V)$ have a common neighbor. So $diam(G(V)) \leq 2$. ■

A consequent of Lemmas 18 and 20 we obtain the following.

Corollary 21 $G(V)$ *is never complete.*

Next we give a characterization of vector spaces whose intersection graphs are bipartite. It is well known that a graph is bipartite if and only if it has no odd cycle.

Theorem 22 $G(V)$ *is bipartite if and only if* $\dim(V) = 2$.

Proof. First by Lemma 18, $G(V)$ is bipartite if $\dim(V) = 2$. Suppose that $\dim(V) \geq 3$. Then V contains at least three linear independent vectors a_1, a_2, a_3. It is obvious that $\langle a_i, a_j \rangle \neq V$ for $1 \leq i, j \leq 3$, $i \neq j$. Now $\langle a_1, a_2 \rangle - \langle a_1, a_3 \rangle - \langle a_2, a_3 \rangle - \langle a_1, a_2 \rangle$ form a cycle on three vertices, and so $G(V)$ is not bipartite. ■

Next we study planarity of $G(V)$.

Lemma 23 *If* $\dim(V) \geq 3$, *then* $G(V)$ *contains a* K_7.

Proof. Let $\dim(V) \geq 3$ and let W be a subspace of V with $\dim(W) = n-3$. Then $\dim(\frac{V}{W}) = 3$. It follows that $\frac{V}{W}$ contains q^2+q+1 subspaces $\frac{W_1}{W}, \frac{W_2}{W}, \ldots, \frac{W_{q^2+q+1}}{W}$, where $\dim(\frac{W_i}{W}) = 2$ for $i = 1, 2, \ldots, q^2+q+1$. Also $\dim(W_i) = n-1$ for $i = 1, 2, \ldots, q^2+q+1$. Let $i, j \in \{1, 2, \ldots, q^2+q+1\}$, and $i \neq j$. Since $\dim(W_i) = \dim(W_j) = n-1$ we obtain $W_i + W_j = V$. By Lemma 5, $\dim(W_i \cap W_j) = n-2$. So $W_i \cap W_j \neq 0$. This means that the subgraph of $G(V)$ induced by $W_1, W_2, \ldots, W_{q^2+q+1}$ is complete. Since $q^2+q+1 \geq 7$, $G(V)$ contains a K_7. ■

As a consequent of Lemmas 18 and 23 we obtain the following characterization.

Theorem 24 $G(V)$ *is planar if and only if* $dim(V) = 2$.

References

[1] M. Atiyah, I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co, Reading, Mass.-London-Don Mills, Ont, 1969.

[2] J. Bosak, *The graphs of semigroups*, in: Theory of Graphs and Application, Academic Press, New York, 1964, 119–125.

[3] I. Chakrabarty, S. Ghosh, T. Mukherjee, and M. Sen, *Intersection graphs of ideals of rings*, Discrete Mathematics 309 (2009), 5381–5392.

[4] B. Cskny, G. Pollk, *The graph of subgroups of a finite group*, Czechoslovak Mathematical Journal 19 (1969), 241–247.

[5] P. Frankl, and R. Graham, *Intersection theorems for vector spaces*, European Journal of Combinatorics 6 (1988), 183–187.

[6] Hoffman, Kenneth and Kunze, Ray, *Linear Algebra* Second edition Prentice-Hall, Inc., Englewood Cliffs, N.J. (1971).

[7] S. Singh, Q. Zameeruddin, *Modern Algebra, third reprint,* Vikas Publishing House Pvt. Ltd., Delhi, 1995.

[8] E. Szpilrajn-Marczewski, *Sur deux proprits des classes d'ensembles*, Fund. Math. 33 (1945), 303–307.

[9] D. West, *Introduction To Graph Theory*, Prentice-Hall of India Pvt. Ltd, 2003.

[10] B. Zelinka, *Intersection graphs of finite abelian groups,* Czechoslovak Math. J. 25 (1975), 171–174.