On graphs of semi-orders

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Abstract

Aigner characterized in terms of forbidden subgraphs all graphs whose line graphs are graphs of semi-orders. We determine all graphs whose line graphs (middle graphs, total graphs, respectively) are graphs of semi-orders.

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1 Introduction

By a graph we mean a simple graph $G = (V, E)$ with vertex set $V(G) = V$ and edge set $E(G) = E$. If v is a vertex of a graph G, then $N_G(v)$ denotes the neighborhood of v in G, that is, the set of vertices adjacent to v. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood.

A binary relation R is called a semi-order on V if and only if for all $x, y, z, w \in V$ the following conditions are satisfied: (a) $\neg xRx$, (b) $(xRy \wedge zRw) \Rightarrow (xRw \vee zRy)$, (c) $(xRy \wedge yRz) \Rightarrow (xRw \vee wRz)$. For a semi-order R on a set V, let $G(R)$ be the undirected graph whose vertices are the elements of V, and in which two vertices u and v are adjacent if and only if uRv or vRu. A graph G is called a graph of a semi-order (SO-graph) if it is isomorphic to the $G(R)$ for some semi-order R. The class of SO-graphs, which was previously studied by Roberts [5, 6] (see also [1]), is a proper subclass of the class of comparability graphs. Also, a graph G is an SO-graph if and only its complement $\overline{G}$ is an indifference graph. The class of SO-graphs constitutes an important interface between graphs and

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semi-orders, both for theoretical investigations on their structural properties, and the development of efficient algorithmic methods for otherwise NP-hard problems on semi-orders and their graphs. They arise naturally in many contexts, such as scheduling, genetics, archeology (see [6]), and have been widely studied.

The following theorem is a characterization of SO-graphs in terms of forbidden induced subgraphs.

Theorem 1 ([5]) *A graph G is an SO-graph if and only if it does not contain any of the graphs $K_3 \cup K_1$, $L(S(K_{1,3}))$ and the complement of C_k (for $k \geq 4$) as an induced subgraph.*

The line graph of a graph G, denoted by $L(G)$, is the intersection graph $\Omega(\overline{E}(G))$ of the family $\overline{E}(G) = \{\{u, v\} \colon uv \in E(G)\}$, that is, $L(G)$ is the graph whose vertices are in one-to-one correspondence with the edges of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. Aigner [1] characterized the graphs whose line graphs are SO-graphs.

Theorem 2 ([1]) *The line graph $L(G)$ is an SO-graph if and only if G contains no $K_3 \cup K_2$, $K_{1,3} \cup K_2$, $2K_{1,2}$, $K_{2,3}$, C_5, or $L(S(K_{1,3}))$ as a subgraph, see Figure 1.*

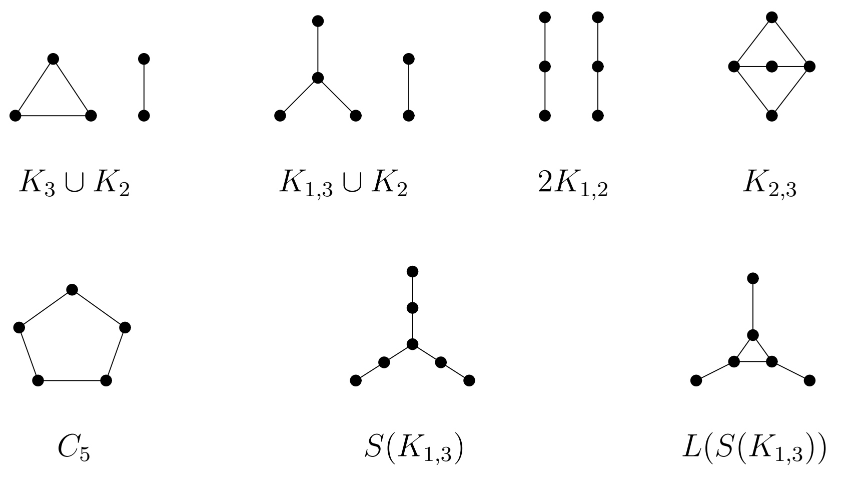

Figure 1

2 Results

Our study of SO-graphs originated with the above result of Aigner. We shall now determine the graphs G for which the line graphs $L(G)$ are SO-graphs.

Theorem 3 *The line graph $L(G)$ is an SO-graph if and only if G is one of the graphs:*

(1) $G = C_4 \cup pK_2 \cup lK_1$;

(2) $G = P_d \cup pK_2 \cup lK_1$, *where* $d \leq 5$;

(3) $G = F \cup lK_1$, *where* l *is a nonnegative integer and* F *is a subgraph of one of the graphs given in Figure 2.*

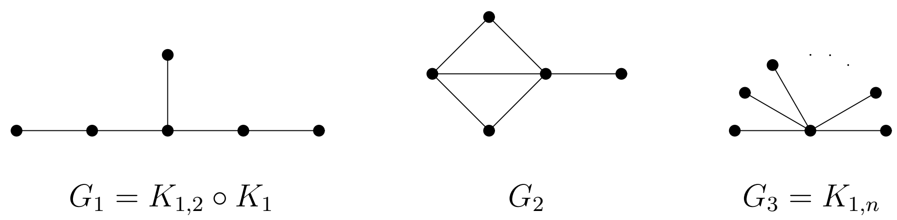

Figure 2

Proof. The necessity follows from Theorem 2. On the other hand, since for every graph F and integer l we have $L(F) = L(F \cup lK_1)$, the family of graphs whose line graphs are SO-graphs is completely determined by the family of graphs without isolated vertices whose line graphs are SO-graphs. Therefore we can confine our considerations only to graphs without isolated vertices. Let F be such a graph. Assume that the line graph $L(F)$ is an SO-graph.

First assume that F is disconnected, and let k be the number of components of F. By Theorem 2, the graph F contains no $2K_{1,2}$. Therefore at most one component of F has at least two edges. Thus either $F = kK_2$ or $F = H \cup (k - 1)K_2$, where H is a connected component with at least two edges. In the first case, $G = F \cup lK_1 = P_2 \cup (k-1)K_2 \cup lK_1$ is of the form (2). In the second case, since F contains no $K_3 \cup K_2$, $K_{1,3} \cup K_2$, $2K_{1,2}$ or C_5, we conclude that H contains no K_3, $K_{1,3}$, $2K_{1,2}$ or C_5. It is easy to observe that H is either C_4 or a path on at most five vertices. Consequently, $G = C_4 \cup (k-1)K_2 \cup lK_1$ or $G = P_d \cup (k-1)K_2 \cup lK_1$, where $d \leq 5$. Thus G is a graph of the form either (1) or (2).

Now assume that the graph F is connected. Let $d = d(F)$ be the diameter of F, and let $P = (v_0, v_1, \ldots, v_d)$ be a diametrical path in F. For a positive integer k, let Y_k be the set of vertices of F at distance k from the path P, i.e., $Y_k = \{x \in V(F) \colon d_F(x, P) = \min_{y \in V(P)} d_F(x, y) = k\}$. By $Y(i)$ we denote the set of vertices which belong to Y_1 and are adjacent to the vertex v_i of P. Note that $Y_1 = \bigcup_{i=0}^{d} Y(i)$. If $d(F) \geq 5$, then the subgraph of F generated by the edges v_0v_1, v_1v_2, v_3v_4 and v_4v_5 is isomorphic to $2K_{1,2}$. Theorem 2 implies that $L(F)$ is not an SO-graph.

Now assume that $d(F) = 4$. If $V(F) = V(P)$, then $F = P = P_5$ and F is a subgraph of G_1 shown in Figure 2. Now assume that $V(F) \neq V(P)$. Thus F is a supergraph of P and $Y_1 \neq \emptyset$. First we claim that $Y(0) = \emptyset$. Otherwise, for $x \in Y(0)$ the subgraph of F generated by the edges xv_0, v_0v_1, v_2v_3 and v_3v_4 is isomorphic to $2K_{1,2}$, a contradiction to Theorem 2. Obviously, we also have

$Y(4) = \emptyset$. Similarly we can show that $Y(1) = Y(3) = \emptyset$. We now conclude that $Y(2) \neq \emptyset$. The impossibility of $K_{1,3} \cup K_2$ in F implies that $|Y(2)| = 1$. From this and the fact that $S(K_{1,3})$ cannot be a subgraph of F it follows that $Y_k = \emptyset$ for $k \geq 2$, and hence F is the graph G_1 shown in Figure 2.

Now assume that $d(F) = 3$. Then either $F = P = P_4$ (and F is a subgraph of G_1 and G_2 shown in Figure 2) or F is a supergraph of P_4. In the second case, the impossibility of $K_{1,3} \cup K_2$ in F implies that each of the sets $Y(i)$ has at most one element. First we prove that some of the sets $Y(0)$ and $Y(3)$ is empty. Suppose that both these sets are nonempty. We have either $Y(0) = Y(3)$ or $Y(0) \neq Y(3)$, and then C_5 or $2K_{1,2}$ is a subgraph of F, a contradiction to Theorem 2. In the case $Y(0) \neq \emptyset$ (and then $Y(3) = \emptyset$), let x be the unique vertex of $Y(0)$. Since $d_F(x, v_3) \leq 3$ and F contains no $K_3 \cup K_2$ or $2K_{1,2}$, it is easy to observe that we must have $Y(0) = Y(2) = \{x\}$, $Y(1) = Y(3) = \emptyset$, and then $Y_k = \emptyset$ for $k \geq 2$. From this we conclude that F is the cycle of length four with one pendant edge, and it is a subgraph of the graph G_2 shown in Figure 2. Now suppose that $Y(0) = Y(3) = \emptyset$ and $Y(1) \cup Y(2) \neq \emptyset$. Since $2K_{1,2}$ is not a subgraph of F, the set $Y(1) \cup Y(2)$ has only one element, say x. Hence we have either $x \notin Y(2)$, $x \notin Y(1)$ or $\{x\} = Y(1) = Y(2)$. Further, we have $Y_k = \emptyset$ for $k \geq 2$. Otherwise $Y_2 \neq \emptyset$ and $xy \in E(F)$ for $y \in Y_2$. But now either $d_F(y, v_3) = d_F(y, v_0) = 4 > d(F)$, or $L(S(K_{1,3}))$ is a subgraph of F, a contradiction. From this it follows that F is a graph obtained from a path P_4 by attaching a new vertex and joining it to one or two inner vertices of P_4. Finally, note that in each case the graph F is a subgraph of the graph G_2 shown in Figure 2.

Now assume that $d(F) = 2$. The result is obvious if $F = P = P_3$. In the case $F \neq P_3$, since $K_{1,3} \cup K_2$ cannot be a subgraph of F, we have $|Y(0)| \leq 2$ and $|Y(2)| \leq 2$. If $|Y(0)| = 2$, say $Y(0) = \{x, y\}$, then, since none of $K_3 \cup K_2$, $K_{1,3} \cup K_2$ and $K_{2,3}$ is a subgraph of F, it must be $xy \notin E(F)$ and $Y(2) \not\subset Y(0)$. Hence we have either $Y(2) = \emptyset$, or $|Y(2)| = 1$, say $Y(2) = \{y\}$. In the first case, since $d_F(x, v_2) = d(F)$ and $d_F(y, v_2) = d(F)$, we have $xv_1, yv_1 \in E(F)$. Then we easily obtain $Y(1) = Y(0)$. Therefore F contains an induced subgraph isomorphic to G_2. Since F contains no $2K_{1,2}$, the graph G_2 cannot be a proper induced subgraph of F. Hence $F = G_2$. We now show that the second case cannot occur. Otherwise, similarly as in the first case, we have $xv_1 \in E(F)$. Then $K_3 \cup K_2$ is a subgraph of F, a contradiction. Now suppose that $|Y(0)| = |Y(2)| = 1$. First we show that, if $Y(0) = \{x\}$ and $Y(2) = \{y\}$, then $x = y$. Suppose the contrary. Since $d_F(x, v_2) = d(F)$, we get $xv_1 \in E(F)$ or $xy \in E(F)$. Hence $K_3 \cup K_2$ or C_5 is a subgraph of F, a contradiction. Therefore we get $Y(0) = Y(2)$. In respect that $K_{1,3} \cup K_2$ cannot be a subgraph of F, we must have $|Y(1)| \leq 2$. If $|Y(1)| = 2$, then, by the impossibility of $2K_{1,2}$ in F we have $Y(0) = Y(2) \subset Y(1)$, $Y_2 = \emptyset$, and therefore $F = G_2$. In the case $|Y(1)| \leq 1$, by the same arguments we get that F is a subgraph of G_2. If $Y(2) = \emptyset$ and $Y(0)$ has only one element, say x, then, since $d_F(x, v_2) = d(F)$, it must be $xv_1 \in E(F)$, and therefore $Y(0) \subset Y(1)$.

From the absence of $K_{1,3} \cup K_2$ in F it follows that $|Y(1)| \in \{1, 2\}$. In the case $|Y(1)| = 1$ we have $Y_1 = \{x\}$ and $Y_2 = \emptyset$. Thus F is a cycle of length three with one pendant edge, and it is a subgraph of G_2. If $|Y(1)| = 2$, then $Y_1 = Y(1)$, $Y_2 = \emptyset$, and therefore F is a subgraph of G_2. In the case $Y(0) = Y(2) = \emptyset$ we have $Y(1) = Y_1 \neq \emptyset$. First observe that $Y_2 = \emptyset$. Otherwise, for $y \in Y_2$, any shortest path joining y and v_0 must contain the vertex v_1 and some vertex of Y_1. Therefore $d_F(y, v_0) = 3 > d(F)$, a contradiction. If no two vertices of $Y(1)$ are adjacent, then $F = G_3 = K_{1,n}$ for $n = |Y(1)| + 2$. If some vertices of $Y(1)$ are adjacent, then from the absence of $K_{1,3} \cup K_2$ in F it follows that $|Y(1)| = 2$ and F is a cycle C of length three with two pendant edges incident to the same vertex of C. Hence, F is a subgraph of G_2.

If $d(F) = 1$, then F is a complete graph K_n. Since C_5 cannot be a subgraph of F, we get $F = K_n$ for some $n \leq 4$. ∎

The following result follows from Theorems 1 and 3.

Corollary 4 *Let H be a connected component of a graph G. If the line graph $L(G)$ is an SO-graph, then H is either an SO-graph, the path P_5, or the graph G_1 given in Figure 2.*

The middle graph $M(G)$ of a graph G is defined to be the intersection graph $\Omega(F)$ of the family $F = \overline{V}(G) \cup \overline{E}(G) = \{\{v\} : v \in V(G)\} \cup \{\{u, v\} : uv \in E(G)\}$. It is known that $M(G)$ is isomorphic to the line graph $L(G \circ K_1)$ of the corona $G \circ K_1$ of G and K_1, see [4]. (The graph $G \circ K_1$ is a graph obtained by taking G and $|V(G)|$ copies of K_1, and joining the i-th vertex of G to the i-th copy of K_1.)

We now determine graphs G for which the middle graph $M(G)$ is an SO-graph.

Corollary 5 *The middle graph $M(G)$ is an SO-graph if and only if G is either K_2, $K_2 \cup pK_1$ or pK_1, where p is an arbitrary nonnegative integer.*

Proof. First assume that G is one of the graphs K_2, $K_2 \cup pK_1$ or pK_1. Then $G \circ K_1$ is one of $G_1 = K_{1,2} \circ K_1 = K_2 \circ K_1$, $P_3 \cup pK_2$, or pK_2. By Theorem 3, the middle graph $M(G) = L(G \circ K_1)$ is an SO-graph.

Now assume that $M(G) = L(G \circ K_1)$ is an SO-graph. By Theorem 2, the graph $G \circ K_1$ contains no $K_3 \circ K_1$, $2K_{1,2}$ or $S(K_{1,3})$. Therefore G does not contain any cycle, and it must be $\Delta(G) \leq 2$. This implies that G is a union of disjoint paths. Now it is easy to verify that the only possible graphs G are K_2, $K_2 \cup pK_1$ and pK_1. ∎

The total graph of a graph G, denoted by $T(G)$, is the intersection graph $\Omega(F)$ of the family $F = \overline{E}(G) \cup \overline{VE}(G) = \{\{u, v\} : uv \in E(G)\} \cup \{\{u\} \cup \{\{u, v\} : v \in N_G(u)\} : u \in V(G)\}$, that is, $T(G)$ is the graph for which there exists a one-to-one correspondence between its vertices and the vertices and edges of G such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements in G are adjacent or incident. This concept was originated by Behzad [3]. It is interesting to note that the graphs G and $L(G)$ are induced subgraphs of the total graph $T(G)$.

We now determine graphs G for which the total graph $T(G)$ is an SO-graph.

Theorem 6 *The total graph $T(G)$ of a graph G is an SO-graph if and only if G is one of the graphs K_2, $K_{1,2}$, K_3 or nK_1 for an arbitrary nonnegative integer n.*

Proof. Since none of induced subgraphs of the total graphs $T(nK_1)$, $T(K_2)$, $T(K_{1,2})$, $T(K_3)$ (see Figure 3) is isomorphic to any of the forbidden subgraphs enumerated in Theorem 1, the graphs $T(nK_1)$, $T(K_2)$, $T(K_{1,2})$, $T(K_3)$ are SO-graphs.

Now assume that $T(G)$ is an SO-graph and $G \neq nK_1$. First we claim that every two edges vu and wt of G are adjacent. Otherwise, the subgraph induced by the vertices $v, u, \{v, u\}$ and $\{w, t\}$ in $T(G)$ is isomorphic to $K_3 \cup K_1$, a contradiction to Theorem 1. Next we show that every vertex of G is incident or adjacent to every edge of G. Suppose not, and let v be a vertex of G which is neither incident or adjacent to any edge uw of G. Then the subgraph induced by the vertices v, u, w and $\{u, w\}$ in $T(G)$ is isomorphic to $K_3 \cup K_1$, a contradiction.

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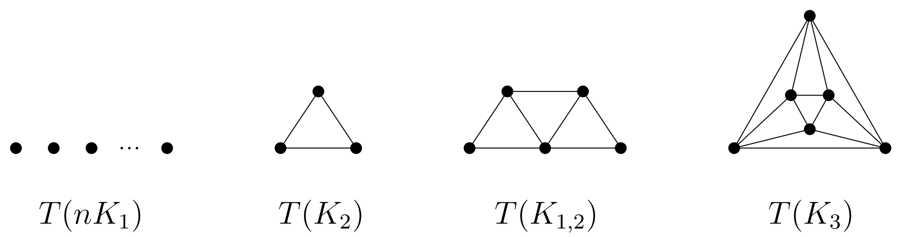

Figure 3

3 Remarks

A graph G is called to be uniquely semi-orderable (USO) if G is an SO-graph and if P and Q are two relations such that $G(P) = G(Q)$, then $P = Q$ or $P = Q^{-1}$, where Q^{-1} denotes the dual of Q. Combining the results of Theorem 3 (Corollary 5 and Theorem 6, respectively) and the characterization of USO-graphs [[1], Theorem 17; [2], Theorem 3] we can easily characterize graphs whose line graphs (middle graphs, total graphs, respectively) are USO-graphs.

Problems considered in the previous section can be reformulated in terms of graph equations of type:

(1) $L(G) = H$,

(2) $M(G) = H,$

(3) $T(G) = H,$

with restriction on H to be an SO-graph. Obviously, the complete solution of (1), (2) and (3) can be deduced from Theorem 3, Corollary 5 and Theorem 6, respectively.

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