Linear programming for determining total dominating sets in graphs

Marcin Krzywkowski[*†]
marcin.krzywkowski@gmail.com

D. A. Mojdeh[‡] and B. Yazdanmehr[‡]
damojdeh@ipm.ir

Abstract

One of the most important problems in graph theory is to find the minimum values of domination parameters. The problems are usually NP-hard. We study total domination and total dominating polynomials from viewpoint of linear programming.
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1 Introduction

In graph theory, to find the edge cover, vertex cover or dominating set of minimum cardinality, as well as independent set and matching of maximum cardinality, and their polynomials, are important and applied concepts. There exist algorithms for finding some of them, but the problems are NP-hard. We study the concept of total dominating sets and total dominating polynomials from viewpoint of linear and binary programming. We compute the coefficients of the polynomials by solving a system of linear equations with $\{0, 1\}$ variables.

[*]Research fellow at the Department of Mathematics, University of Johannesburg, South Africa.
[†]Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Poland. Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.
[‡]Department of Mathematics, University of Tafresh, P.O. Box 39518-79611, Tafresh, Iran

Let $G = (V, E)$ be a graph with $|V(G)| = n$ and $|E(G)| = m$. Then the vertex cover and vertex dominating polynomials are of degree n, the edge cover and edge dominating polynomials are of degree m, and the coefficient of x^k is the number of such sets of cardinality k. Also the matching and independence polynomials are of degree at most n such that the coefficient of x^k is the number of matchings and independent sets of cardinality k, respectively. Graph polynomials, including dominating, edge and vertex covering, independence, and matching have been studied for example in [1, 2, 4–6].

In a graph G, a subset $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A subset $D \subseteq V(G)$ is a total dominating set of G if every vertex of G has a neighbor in D. The minimum cardinality of a total dominating set of G is the total domination number of G, and is denoted by $\gamma_t(G)$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. The total dominating polynomials are of degree n, and the coefficient of x^k is the number of total dominating sets of cardinality k.

We denote the adjacency matrix by A, where $A = [a_{ij}]_{n\times n}$ and a_{ij} is the number of edges with endpoints v_i and v_j. A complete graph K_n is a graph whose vertices are pairwise adjacent. A graph G is bipartite if $V(G)$ is a union of two disjoint (possibly empty) independent sets called partite sets of G. A complete bipartite graph is a bipartite graph in which two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes p and q, the graph is denoted by $K_{p,q}$. The complement $\overline{G}$ of a graph G is a graph with vertex set $V(G)$ such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The join $G_1 \vee G_2$ of two vertex-disjoint graphs G_1 and G_2, is a graph with $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy\colon x \in V(G_1), y \in V(G_2)\}$.

2 Algorithm for total domination

Using linear programming on adjacency matrix, we have the following results for determining the minimum size of a total dominating set. For more details on linear programming, see [3].

Let $V(G) = (v_1, v_2, \ldots, v_n)^t$ and $1_n = (1, 1, \ldots, 1)^t$.

Theorem 1 *Let A be the adjacency matrix of a connected graph $G = (V, E)$ with $|V(G)| = n$. Then the minimum size of a total dominating set can be obtained as follows,*

$$\gamma_t = min \sum_{j=1}^{n} v_j$$

$$\textit{subject to: } (A + I_n)V \geq 1_n$$

$$v_j \in \{0, 1\} \textit{ where } j = 1, 2, \ldots, n$$

$$v_j = 1 \Rightarrow \exists v_i \in N_G(v_j) \text{ such that } v_i = 1.$$

Proof. Let D be a total dominating set of G. Since we want to obtain a smallest total dominating set, we have a minimization problem, where the object function is $\gamma_t = \min \sum_{j=1}^{n} v_j$. On the other hand, for each $v_j \in V(G)$, at least one of the vertices $v_k \in D$ must be adjacent to v_j. Thus from every row of $A + I_n$, at least one entry (v_k) must be equal to 1. Also, D has no isolated vertex. Therefore

$$a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n \geq 1$$

$$a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n \geq 1$$

$$\vdots$$

$$a_{n1}v_1 + a_{n2}v_2 + \ldots + a_{nn}v_n \geq 1$$

$$v_j \in \{0, 1\} \text{ where } j = 1, 2, \ldots, n$$

$$v_j = 1 \Rightarrow \exists v_i \in N_G(v_j) \text{ such that } v_i = 1.$$

■

Definition 2 *A total dominating polynomial $TD(G, x)$ or $TD(x)$ is as follows:*

$$TD(x) = f_{\gamma_t} x^{\gamma_t} + f_{\gamma_{t+1}} x^{\gamma_{t+1}} + \ldots + f_n x^n,$$

where γ_t is the total domination number and f_is are the numbers of total dominating sets of cardinality i.

Theorem 3 *Let A be the adjacency matrix of a connected graph $G = (V, E)$ with $|V(G)| = n$. Then the number of total dominating sets of cardinality k, where $\gamma_t \leq k \leq n$, can be obtained as follows,*

$$\sum_{j=1}^{n} v_j = k \tag{1}$$

$$\textit{subject to: } (A + I_n)V \geq 1_n \tag{2}$$

$$v_j \in \{0, 1\} \textit{ where } j = 1, 2, \ldots, n$$

$$v_j = 1 \Rightarrow \exists v_i \in N_G(v_j) \textit{ such that } v_i = 1. \tag{3}$$

Proof. The inequality (2) and the phrase (3) are the conditions for a set to be total dominating, and the equality (1) for each k is satisfied when we have a total dominating set of cardinality $\gamma_t, \gamma_{t+1}, \ldots, n$, respectively, and with this process we can compute $f_{\gamma_t}, f_{\gamma_{t+1}}, \ldots, f_n$. It is trivial that $f_n = 1$. ■

Theorem 4 *Let $G = (V, E)$ be a connected graph in which $|V(G)| = n$. Then:*

(a) If G has no leaf, then $f_{n-1} = n$.

(b) $f_n = 1$.

(c) $f_i = 0$ if and only if $i > n$ or $i < \gamma_t(G)$.

(d) $TD(x)$ has no constant term.

(e) $TD(x)$ is strictly increasing on $[0, \infty]$.

(f) If H is a subgraph of G, then deg $TD(H, x) \leq$ deg $TD(G, x)$.

(g) Zero is a root of $TD(x)$ with respect to $\gamma_t(G)$.

Proof. We only prove (a) and (g). The remaining statements are straightforward.

(a) By discarding any vertex we obtain a total dominating set. Thus there are n total dominating sets of cardinality $n - 1$.

(g) $TD(x) = f_{\gamma_t}x^{\gamma_t} + f_{\gamma_{t+1}}x^{\gamma_{t+1}} \ldots + f_n x^n = x^{\gamma_t} \cdot (f_{\gamma_t} + f_{\gamma_{t+1}}x + \ldots + f_n x^{n-\gamma_t}) = 0$

$$\Rightarrow \begin{cases} x^{\gamma_t} = 0 \text{ or} \\ f_{\gamma_t} + \ldots + f_n x^{n-\gamma_t} = 0 \end{cases}$$ ■

Example 5 *Consider the Petersen graph:*

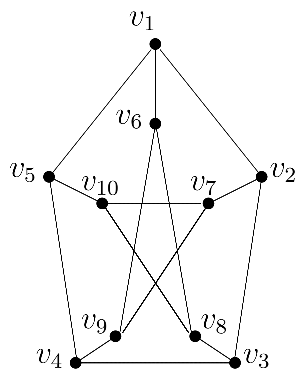

Figure 1

Since $\gamma_t = 4$,

$$TD(x) = f_{\gamma_t}x^{\gamma_t} + f_{\gamma_{t+1}}x^{\gamma_{t+1}} + \ldots + f_n x^n = f_4x^4 + f_5x^5 + \ldots + f_{10}x^{10}$$

is the total dominating polynomial for the Petersen graph. We want to compute these coefficients. Let A_p be the adjacency matrix. Next we solve the problem by using the following algorithm*:*

$$\sum_{j=1}^{10} v_j = k, \ where\ 4 \leq k \leq 10$$

$$(A_p + I_{10})V \geq 1_{10} \Rightarrow$$

$$\begin{array}{cc} v_1 + v_2 + v_5 + v_6 \geq 1 & v_1 + v_2 + v_3 + v_7 \geq 1 \\ v_2 + v_3 + v_4 + v_8 \geq 1 & v_3 + v_4 + v_5 + v_9 \geq 1 \\ v_1 + v_4 + v_5 + v_{10} \geq 1 & v_1 + v_6 + v_8 + v_9 \geq 1 \\ v_2 + v_8 + v_9 + v_{10} \geq 1 & v_3 + v_6 + v_8 + v_{10} \geq 1 \\ v_4 + v_6 + v_7 + v_9 \geq 1 & v_5 + v_7 + v_8 + v_{10} \geq 1 \end{array}$$

$$v_j \in \{0,1\}\ where\ j = 1, 2, \ldots, 10$$

$$v_j = 1 \Rightarrow \exists v_i \in N_G(v_j)\ such\ that\ v_i = 1.$$

First we compute f_4. As we show in Figure 1, we may select three vertices from outside the cycle and one vertex from inside the cycle, or vice versa. Therefore $f_4 = 10$.

If we want to choose five vertices either from outside or from inside the cycle, then obviously there are 2 ways of doing this.

Now assume that we choose four vertices from outside the cycle and one vertex from inside the cycle, or vice versa. There are $2 \cdot 5 \cdot 2 = 20$ ways of doing this.

Now assume that we choose three vertices from outside the cycle and two vertices from inside the cycle, or vice versa. There are $2 \cdot 5 \cdot 6 = 60$ ways of doing this.

We now conclude that $f_5 = 82$.

Similarly, one can obtain $f_6 = 200$, $f_7 = 118$, $f_8 = 70$, $f_9 = 10$ and $f_{10} = 1$, and the total dominating polynomial of the Petersen graph is:

$$TD(x) = 10x^4 + 82x^5 + 200x^6 + 118x^7 + 70x^8 + 10x^9 + x^{10}.$$

Theorem 6 *Let $G_1, G_2, \ldots, G_m$ be connected components of G such that none of them is an isolated vertex. Then*

$$TD(G, x) = TD(G_1, x) \cdot TD(G_2, x) \cdot \ldots \cdot TD(G_m, x).$$

Proof. It suffices to prove the result for $m = 2$. For $k \geq \gamma_t(G)$, a total dominating set of cardinality k can be obtained by selecting a total dominating set of cardinality j for G_1 (where $\gamma_t(G_1) \leq j \leq |V(G_1)|$) and a total dominating set of cardinality $k - j$ for G_2 (where $\gamma_t(G_2) \leq k - j \leq |V(G_2)|$). The coefficient of x^k in $TD(G_1, x) \cdot TD(G_2, x)$ is obtained by products of the coefficients of x^j and x^{k-j}. ■

3 Coefficients of the total dominating polynomial

We now determine the coefficients of the total dominating polynomial for some families of graphs.

3.1 Coefficients of the polynomial for K_n

Theorem 7 *If $G = K_n$, then $f_i = \binom{n}{i}$, where $n \geq 2$ and $i \geq 2$.*

Proof. We have $\gamma_t = 2$ for complete graphs on at least two vertices. Since all vertices of a complete graph are pairwise adjacent, there are $\binom{n}{i}$ total dominating sets of cardinality i. ■

We now compute the total dominating polynomial for K_n.

Theorem 8 *For every positive integer n we have $TD(K_n, x) = (1+x)^n - 1 - nx$.*

Proof. We have $\gamma_t(K_n) = 2$ and $td(K_n, i) = f_i = \binom{n}{i}$ for $2 \leq i \leq n$. We now get $TD(K_n, x) = \sum_{i=2}^{n} \binom{n}{i} x^i = (1+x)^n - \binom{n}{0} \cdot x^0 - \binom{n}{1} \cdot x^1 = (1+x)^n - 1 - nx$. ■

3.2 Coefficients of the polynomial for P_n

It is well known [7] that the total domination number of a path P_n is:

$$\gamma_t(P_n) = \begin{cases} n/2 & \text{if } n = 4k; \\ \lfloor n/2 \rfloor + 1 & \text{otherwise.} \end{cases}$$

We now give a formula for the number of minimum total dominating sets of paths.

Theorem 9 *For paths P_k we have*

$$f_{\gamma_t} = \begin{cases} 1 & \text{if } n = 4k; \\ k+1 & \text{if } n = 4k-1; \\ k^2 & \text{if } n = 4k-2; \\ k-1 & \text{if } n = 4k-3. \end{cases}$$

Proof. Let D be a $\gamma_t(P_n)$-set. If $n = 4k$, then $\gamma_t(P_n) = 2k$. Since $\{v_2, v_3, v_6, v_7, \ldots, v_{n-2}, v_{n-1}\}$ is the only minimum total dominating set of P_n, we have $f_{\gamma_t}(P_n) = 1$ for $n = 4k$.

Now assume that $n = 4k - 1$. Then $\gamma_t(P_n) = \lfloor n/2 \rfloor + 1 = 2k$. We have the following facts.

Fact 1. Assume that $v_1, v_n \notin D$. Then v_2, v_3, v_{n-2} and v_{n-1} should be in D. For $4k-7$ other vertices of P_n, $2k-4$ of them should be in D. Since at most two consecutive vertices can be outside the set D, there is a vertex v_i ($4 \leq i \leq n-3$) such that $v_i \notin D$, while $v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2} \in D$. Choosing v_i leads to $k-1$ total dominating sets.

Fact 2. Assume that $v_1 \notin D$ and $v_2, v_3, v_{n-1}, v_n \in D$. Then from the $4k-6$ remaining vertices, $2k-4$ should be in D. Then the other vertices of D form the set $\{v_{4i+2}, v_{4i+3} \colon 1 \leq i \leq k-2\}$. Thus there is only one such total dominating set for P_n.

Fact 3. Assume that $v_n \notin D$ and $v_1, v_2, v_{n-2}, v_{n-1} \in D$. Similarly as in Fact 2, there is only one such total dominating set for P_n.

Fact 4. The case $v_1, v_2, v_{n-1}, v_n \in D$ is not possible, since any choice gives us three consecutive vertices not being in D. We now conclude that $f_{\gamma_t}(P_n) = k+1$ if $n = 4k-1$.

Now assume that $n = 4k-3$. Then $\gamma_t(P_n) = \lfloor n/2 \rfloor + 1 = 2k-1$. No total dominating set includes both vertices v_1 and v_2, or both vertices v_{n-1} and v_n, as then there are three consecutive vertices which are not in D. Assume that $v_1, v_n \notin D$. Then the vertices $v_2, v_3, v_{n-2}, v_{n-1}$ should be in D. Also for the $4k-9$ other vertices, $2k-5$ vertices should be in D. In this case we have to choose these $2k-5$ vertices in such a way that only one vertex from each set $\{v_{4i-2}, v_{4i-1}, v_{4i}\}$ ($1 \leq i \leq k-1$) belongs to D. Thus $f_{\gamma_t}(P_n) = k-1$ if $n = 4k-3$.

Now assume that $n = 4k-2$. Then $\gamma_t(P_n) = \lfloor n/2 \rfloor + 1 = 2k$. We have the following facts.

Fact 1. The vertices are divided into two sets of 5 consecutive vertices and $k-3$ sets of 4 consecutive vertices. The placement of 5-vertex sets implies that there are $\binom{k-1}{2} = (k-1)!/(2!(k-3)!) = (k-1)(k-2)/2$ such total dominating sets.

Fact 2. The vertices are divided into one set of 6 consecutive vertices and $k-2$ sets of 4 consecutive vertices. The placement of the 6-vertex set implies that there are $k-1$ such total dominating sets.

Fact 3. Assume that $v_1, v_n, v_4, v_{n-3} \notin D$ and $v_2, v_3, v_{n-1}, v_{n-2} \in D$. Then from the $4k-10$ remaining vertices, $2k-4$ should be in D.

If $v_5, v_6, v_{n-4}, v_{n-5} \in D$, then there is only on way for choosing the total dominating set.

If $v_5 \notin D$ and $v_{n-4} \in D$, then $v_{n-5} \in D$ and there exists a vertex v_{4i} such that $v_{4i} \notin D$ and $v_{4i-1}, v_{4i-2}, v_{4i+1}, v_{4i+2} \in D$ for $2 \leq i \leq k-2$. Thus the placement of the vertex v_{4i} implies that there are $k-3$ such total dominating sets.

If $v_5 \in D$ and $v_{n-4} \notin D$, then $v_6 \in D$ and similarly as in the previous possibility, there exist $k-3$ such total dominating sets.

If $v_5, v_{n-4} \notin D$, then for the $4k-12$ remaining vertices, $2k-4$ should be in D and $2k-8$ should be outside D. Then there exist two vertices v_i and v_j in $P_n \setminus D$, where $8 \leq j \leq i \leq n-7$ and $i-j \geq 3$. Placement of the vertices implies that there are $\binom{k-3}{2} = (k-3)!/(2!(k-5)!) = (k-3)(k-4)/2$ such total dominating sets.

Therefore Fact 3 gives us $1+2(k-3)+(k-3)(k-4)/2$ total dominating sets.

Fact 4. Now assume that $v_1, v_2 \in D$ and $v_3, v_n, v_{n-3} \notin D$. Then $v_{n-1}, v_{n-2} \in D$. Thus from the $4k-9$ remaining vertices, $2k-4$ should be in D and $2k-5$ should be outside D.

If $v_4 \notin D$, then there exists only one total dominating set in which $v_{4i+1}, v_{4i+2} \in D$ (for $1 \leq i \leq k-2$) and $v_{4i-1}, v_{4i} \notin D$ (for $2 \leq i \leq k-2$).

If $v_4 \notin D$, then similarly as in the previous possibility, there is only one such total dominating set.

If $v_4, v_{n-4} \notin D$, then total dominating sets should be chosen in such a way that exactly one vertex from the set $\{v_{4i+3} \colon 1 \leq i \leq k-3\}$ does not belong to D. Thus there are $k-3$ total dominating sets.

We now conclude that in this fact we obtain $k-1$ total dominating sets.

Fact 5. Now assume that $v_1, v_4, v_{n-2} \notin D$. Then $v_2, v_3, v_n, v_{n-1} \in D$. Similarly as in the previous fact, there are $k-1$ such total dominating sets.

Fact 6. Now assume that $v_1, v_2, v_n, v_{n-1} \in D$. From the $4k-6$ remaining vertices, $2k-4$ should be in D and $2k-2$ should be outside D. Thus there is only one total dominating set, as the vertices of $\{v_{4i+1}, v_{4i+2} \colon 1 \leq i \leq k-2\}$ should be in D and the vertices of $\{v_{4i-1}, v_{4i} \colon\ 1 \leq i \leq k-1\}$ should be outside D.

We now conclude that the number of minimum total dominating sets for P_{4k-2} is

$$\frac{(k-1)(k-2)}{2} + (k-1) + 1 + 2(k-3) + \frac{(k-3)(k-4)}{2} + 2(k-1) + 1 = k^2.$$

■

Let us observe that for any path P_n, if a vertex v_i is outside a total dominating set, then $i \neq 2$ or $i \neq n-1$.

Observation 10 *We have $td(P_n, n-1) = n-2$ for $n \geq 4$.*

We now find the number of total dominating sets of a path P_n of cardinality $n-2$.

Theorem 11 *For each positive integer $n \geq 6$ we have*

$$td(P_n, n-2) = 1 + \frac{(n-4)(n-3)}{2}.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n, and let D be a total dominating set of cardinality $n-2$. If $v_1, v_n \notin D$, then there is only one such total dominating set. Now assume that $v_1 \notin D$ and $v_n \in D$. Then $v_2, v_3, v_{n-1} \in D$. The other vertex outside the set D is one of the vertices of the set $\{v_i \colon 4 \leq i \leq n-2\}$. There are $n-5$ such total dominating sets. Now assume that $v_n \notin D$ and $v_1 \in D$. Then

$v_2, v_{n-1}, v_{n-2} \in D$. Similarly as in the previous possibility, there are $n-5$ such total dominating sets. Now assume that $v_1, v_2, \ldots, v_k, v_n, v_{n-1}, \ldots, v_{n-(k-1)} \in D$ for $2 \leq k \leq \lfloor (n-4)/2 \rfloor$. If $v_{k+1} \notin D$, then the other vertex outside D is one of the vertices of the set $\{v_i \colon k+4 \leq i \leq n-k\}$. There exist $n-(2k+3)$ such total dominating sets. If $v_{n-k} \notin D$, then the other vertex outside D is one of the vertices of the set $\{v_i \colon k+5 \leq i \leq n-k\}$. There exist $n-(2k+4)$ total dominating sets with $n-2$ vertices. Thus there are $1+2+\ldots+(n-8)+(n-7) = (n-7)(n-6)/2$ such total dominating sets. We now conclude that the number of total dominating sets of P_n of cardinality $n-2$ equals $1+3(n-5)+(n-7)(n-6)/2 = 1+(n-6)+(n-5)+(n-4)+(n-7)(n-6)/2 = 1+(n-4)(n-3)/2$. ■

Since $(n-4)(n-3)/2 = (n-5)(n-4)/2+n-4$, we get the following corollary from Observations 10 and 11.

Corollary 12 *For each positive integer $n \geq 7$ we have*

$$td(P_n, n-2) = td(P_{n-1}, n-3) + td(P_{n-2}, n-3).$$

Theorem 13 *For every positive integer $n \geq 6$ we have $td(P_n, n-2) = \binom{n}{n-2} -2-3(n-3)$.*

Proof. The result we prove by the induction on the number n. For P_6, Theorem 11 implies that $td(P_6, 4) = 1+(6-4)(6-3)/2 = 4 = \binom{6}{6-2} - 2 - 3 \cdot (6-3)$. Assume that the result is true for P_k, where $6 \leq k \leq n-1$. Let $k = n$ and consider a path P_n. By Theorem 11 we have $td(P_n, n-2) = td(P_{n-1}, n-3) + td(P_{n-2}, n-3)$. Applying the inductive hypothesis and Observation 10 we get $td(P_{n-1}, n-3) = \binom{n-1}{n-3} - 2 - 3(n-4) + (n-4) = (n-1)(n-2)/2 - 2 - 2(n-4) = (n^2-7n+14)/2 = n(n-1)/2 - 3(n-3) - 2 = \binom{n}{n-2} - 2 - 3(n-3) = td(P_n, n-2)$. ■

3.3 Coefficients of the polynomial for C_n

It is well known [7] that the total domination number of a path C_n is:

$$\gamma_t(C_n) = \begin{cases} n/2 & \text{if } n = 4k; \\ \lfloor n/2 \rfloor + 1 & \text{otherwise.} \end{cases}$$

We now give a formula for the number of minimum total dominating sets of cycles.

Theorem 14 *For paths P_n we have*

$$f_{\gamma_t} = \begin{cases} n & \textit{if } n = 4k-1; \\ 4 & \textit{if } n = 4k; \\ n & \textit{if } n = 4k+1; \\ n^2/4 & \textit{if } n = 4k+2. \end{cases}$$

Proof. Let D be a $\gamma_t(C_n)$-set. First assume that $n = 4k-1$. Then $\gamma_t = (C_n) = 2k$. There is a vertex v_i outside D such that $v_{i+1}, v_{i+2}, v_{i-1}, v_{i-2}$ (mod n) should be in D. Changing i between 1 to n gives us n total dominating sets for C_n. Thus $f_{\gamma_t}(C_{4k-1}) = 4k-1$.

Now assume that $n = 4k$. Then $\gamma_t = (C_n) = 2k$. If $v_1, v_2 \in D$, then $D = \{v_{4i+1}, v_{4i+2} \colon 0 \le i \le k-1\}$. If $v_2, v_3 \in D$, then $D = \{v_{4i+2}, v_{4i+3} \colon 0 \le i \le k-1\}$. If $v_3, v_4 \in D$, then $D = \{v_{4i+3}, v_{4i+4} \colon 0 \le i \le k-1\}$. If $v_4, v_5 \in D$, then $D = \{v_{4i+4}, v_{4i+5} \colon 0 \le i \le k-1\}$.

The remaining cases are similar. Therefore $f_{\gamma_t}(C_{4k}) = 4$.

Now assume that $n = 4k+1$. Then $\gamma_t(C_n) = 2k+1$. The set D contains three consecutive vertices v_1, v_2, v_3 and the other vertices of D are vertices of the set $\{v_{4i+2}, v_{4i+3} \colon 1 \le i \le k-1\}$. There are n different triples of consecutive vertices in C_n. Thus there are n total dominating sets. Consequently, $f_{\gamma_t}(C_{4k+1}) = 4k+1$.

Now assume that $n = 4k+2$. Then $\gamma_t = (C_n) = 2k+2$. The total dominating sets of C_n are determined as follows.

Fact 1. There are four consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ in D. Thus the other vertices of D are $v_{i+6}, v_{i+7}, v_{i+10}, v_{i+11}, \ldots, v_{i-4}, v_{i-3}$. Since i can take all indices, thus there are $4k+2$ such total dominating sets.

Fact 2. If $v_i, v_{i+1}, v_{i+2}, v_j, v_{j+1}, v_{j+2}$ (where $j-i \ge 5$ and $|j+2-i| \ge 3$) are in D, then the other vertices of D are two consecutive vertices. Without loss of generality we assume that $i = 1$. If k is even, then for every j such that $j \in \{4t+2 \colon 1 \le t \le k/2-1\}$ there are $4k+2$ total dominating sets. For $j = 2k$ there are $2k+1$ total dominating sets. The other js give us repeated total dominating sets. Thus there are $k(4k+2)/2 - (2k-1)$ total dominating sets. If k is odd, then for every j such that $j \in \{4t+2 \colon 1 \le t \le (k-1)/2\}$ there are $4k+2$ total dominating sets. The other js give us repeated total dominating set. Thus there are $(k-1)(4k+2)/2$ such total dominating sets.

Fact 3. Let v_i and v_j (where $|j-i| \ge 3 \bmod n$) be two vertices outside D such that $v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}$ and $v_{j-1}, v_{j-2}, v_{j+1}, v_{j+2}$ are in D. Without loss of generality assume that $i = 1$. If k is even, then there are $k/2$ total dominating sets. Thus there are $k(4k+2)/2$ total dominating sets. If k is odd, then there are $(k-1)(4k+2)/2$ total dominating sets, and for $j = 4 \cdot (k+1)/2$ there are $2k+1$ total dominating sets. The other js give us repeated total dominating sets. We now conclude that $f_{\gamma_t}(C_{4k+2}) = (2k+1)^2$. ■

Since discarding each vertex of C_n gives a total dominating set for C_n, we have the following observation.

Observation 15 $td(C_n, n-1) = n$.

We now find the number of total dominating sets of a path C_n of cardinality $n-2$.

Theorem 16 *For every positive integer $n \geq 5$ we have*

$$td(C_n, n-2) = \frac{(n-2)(n-1)}{2} - 1.$$

Proof. Let D be a total dominating of C_n of cardinality $n-2$. If the two vertices outside D are v_i and v_{i+1}, then there are n such total dominating sets. Now assume that the two vertices outside D are v_i and v_j, where $d(v_i, v_j) \geq 3$. If $i = 1$, then $j \in \{4, 5, \ldots, n-2\}$. There are $n-5$ such total dominating sets. Now assume that $i = 2$. Then $j \in \{5, 6, \ldots, n-1\}$. There are $n-5$ such total dominating sets. Now assume that $i = 3$. Then $j \in \{6, 7, \ldots, n\}$. There are $n-5$ total dominating sets. Generally, if $i = k$ where $4 \leq k \leq n-3$, then $j \in \{k+3, k+4, \ldots, n\}$. There are $n-k-2 = (n-5)-(k-3)$ such total dominating sets. We now get $td(C_n, n-2) = n+1+2+\ldots+(n-6)+3(n-5)$ $= n+(n-3)(n-2)/2-3 = (n-2)(n-1)/2-1$. ■

Since $(n-1)(n-3)/2 = (n-3)(n-2)/2+(n-1)$, we have the following corollary from Observations 15 and 16.

Corollary 17 *For every integer $n \geq 6$ we have*

$$td(C_n, n-2) = td(C_{n-1}, n-3) + td(C_{n-2}, n-3).$$

We have the following corollary from Theorem 14.

Corollary 18 *For every positive integer n we have $td(C_{4n}, 2n) = 4$.*

Theorem 19 *For each integer $n \geq 5$ we have $td(C_n, n-2) = \binom{n}{n-2} - n$.*

Proof. The result we prove by the induction on the number n. If $n = 5$, then by Theorem 16 we have $td(C_5, 3) = (5-2)(5-1)/2-1 = \binom{5}{5-2} - 5$. Assume that it is true for C_k $(5 \leq k \leq n-1)$. Let $k = n$ and consider a cycle C_n. By Corollary 17 we have $td(C_n, n-2) = td(C_{n-1}, n-3) + td(C_{n-2}, n-3)$. Using the inductive hypothesis and Theorem 16 we get $td(C_{n-1}, n-3) = \binom{n-1}{n-3} - (n-1)$. By Observation 15 we have $td(C_{n-2}, n-3) = n-2$. Therefore $td(C_{n-1}, n-3)$ $+td(C_{n-2}, n-3) = \binom{n-1}{n-3} - (n-1) + (n-2) = \binom{n}{n-2} - n = td(C_n, n-2)$. ■

4 Total dominating polynomial for $G_1 \vee G_2$

We now obtain total dominating polynomial for a join of graphs.

Theorem 20 *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. The coefficients of the total dominating polynomial for $G_1 \vee G_2$ can be computed as follows:*

$$f_i = td(G_1 \vee G_2, i) = \sum_{j=1}^{i-1} \binom{n_1}{j}\binom{n_2}{i-j} + td(G_1, i) + td(G_2, i)$$

$$\gamma_t(G_1 \vee G_2) = 2 \ ; \quad 2 \leq i \leq n_1 + n_2$$

such that $td(G_1, i)$ and $td(G_2, i)$ are the numbers of i-element total dominating sets of the graphs G_1 and G_2, respectively.

Proof. We find the number of total dominating sets of cardinality i as follows. Clearly, at least one vertex of G_1 or G_2 must be selected. First we select a vertex of G_1 and if it is possible, $i-1$ vertices of G_2; this is probably equal to $\binom{n_1}{1}\binom{n_2}{i-1}$. We can select two vertices of G_1 and $i-2$ vertices of G_2, this is probably equal to $\binom{n_1}{2}\binom{n_2}{i-2}$. Besides that, it is possible that we select all of i vertices of G_1 or G_2. The number of positions for this way is $td(G_1, i)$ or $td(G_2, i)$, respectively. Then generally we get $td(G_1 \vee G_2, i) = \binom{n_1}{1}\binom{n_2}{i-1} + \binom{n_1}{2}\binom{n_2}{i-2} + \ldots + \binom{n_1}{i-1}\binom{n_2}{1} + td(G_1, i)$ $+td(G_2, i)$. If $i < j$, then $\binom{i}{j} = 0$, also for each $i > n_1$; $td(G_1, i) = 0$ and for each $i > n_2$; $td(G_2, i) = 0$. It is remarkable that $td(G_1 \vee G_2, n_1 + n_2) = \binom{n_1}{n_1}\binom{n_2}{n_2} = 1$. ■

Theorem 21 *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. Then*

$$TD(G_1 \vee G_2, x) = ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1) + TD(G_1, x) + TD(G_2, x).$$

Proof.

$$TD(G_1 \vee G_2, x) = \sum_{i=2}^{n_1+n_2} f_i x^i$$

$$= \sum_{i=2}^{n_1+n_2} [\sum_{j=1}^{i-1} \binom{n_1}{j}\binom{n_2}{i-j}] x^i + TD(G_1, x) + TD(G_2, x)$$

$$= [\sum_{j=1}^{n_1} \binom{n_1}{j} x^j][\sum_{k=1}^{n_2} \binom{n_2}{k} x^k] + TD(G_1, x) + TD(G_2, x)$$

$$= ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1) + TD(G_1, x) + TD(G_2, x).$$

■

Theorem 22 *Let G be a complete bipartite graph $K_{n,m}$. Then*

$$td(K_{n,m}, i) = f_i = \sum_{j=1}^{i-1} \binom{n}{j} \binom{m}{i-j}$$

and

$$TD(K_{n,m}, x) = ((1+x)^n - 1)((1+x)^m - 1) + x^n + x^m.$$

Proof. By the previous theorem, it suffices to consider $G_1 = \overline{K_n}$ and $G_2 = \overline{K_m}$. ■

Corollary 23 *For stars we have*

$$\begin{aligned} TD(K_{n-1,1}, x) &= ((1+x)^{n-1} - 1)x + TD(\overline{K_{n-1}}, x) + TD(K_1, x) \\ &= ((1+x)^{n-1} - 1)x \end{aligned}$$

and

$$f_i = \binom{n-1}{i-1}.$$

Corollary 24 *For each positive integer $n \geq 2$, the total dominating polynomial for W_n (wheel with n vertices) is computed as follows,*

$$TD(W_n, x) = ((1+x)^{n-1} - 1)x + TD(C_{n-1}, x).$$

Proof. In fact, W_n is $C_{n-1} \vee K_1$. Therefore

$$\begin{aligned} TD(C_{n-1} \vee K_1, x) &= ((1+x)^{n-1} - 1)((1+x) - 1) + TD(C_{n-1}, x) + TD(K_1, x) \\ &= ((1+x)^{n-1} - 1)x + TD(C_{n-1}, x). \end{aligned}$$

■

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