On the restrained Roman domination in graphs

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Abstract

A Roman dominating function (RDF) on a graph G is a function $f\colon V(G) \to \{0,1,2\}$ satisfying the condition that every vertex v for which $f(v)=0$, is adjacent to at least one vertex u for which $f(u)=2$. The weight of a Roman dominating function f is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The Roman domination number of G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. For a given graph, a Roman dominating function $f=(V_0,V_1,V_2)$ is a restrained Roman dominating function (rRDF) if every vertex of V_0 has a neighbor in V_0. The restrained Roman domination number of G, denoted by $\gamma_{rR}(G)$, is the minimum weight of an rRDF on G. We first show that the restrained Roman domination problem is NP-complete. Then we give various bounds and characterizations. Finally we study restrained Roman domination in random graphs.

Keywords: Roman domination, restrained Roman domination, complexity, probabilistic method, random graph.

AMS Subject Classification: 05C05, 05C69.

1 Introduction

Let $G=(V,E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices

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of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. The complete graph on n vertices we denote by K_n. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Double star is a graph obtained from a star by joining a positive number of vertices to one of the leaves.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a restrained dominating set of G if additionally every vertex of $V(G) \setminus D$ has a neighbor in $V(G) \setminus D$. The domination (restrained domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_r(G)$, respectively), is the minimum cardinality of a dominating (restrained dominating, respectively) set of G. Restrained domination in graphs was introduced by Telle and Proskurowski [19], albeit indirectly, as vertex partitioning problem, and was further studied for example in [4–7, 9, 10, 12, 22]. For a comprehensive survey of domination in graphs, see [11].

For a graph G, let $f\colon V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G)\colon f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f\colon V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. Thus we will write $f = (V_0, V_1, V_2)$ to refer to f. A function $f\colon V(G) \to \{0, 1, 2\}$ is a Roman dominating function, or just RDF, if every vertex v for which $f(v) = 0$, is adjacent to at least one vertex u for which $f(u) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0, V_1, V_2)$ is called a $\gamma_R(G)$-function if it is an RDF on G and $f(V(G)) = \gamma_R(G)$. The concept of Roman domination in graphs was introduced by Stewart [18], and further studied for example in [3, 8, 13, 15, 16, 21].

As noted in [3], the idea of considering a Roman dominating function is that the assignments 1 and 2 represent either one or two Roman legions stationed at a given location (vertex v). A nearby location (an adjacent vertex u) is considered to be unsecured if no legions are stationed there (i.e. $f(u) = 0$). One can consider this idea with a further condition that any unsecured location is also adjacent to at least one unsecured location. Pushpam and Padmapriea [17] introduced the concept of restrained Roman domination in graphs. An RDF $f = (V_0, V_1, V_2)$ on a graph G is a restrained Roman dominating function, or just rRDF, on G if every vertex of V_0 has a neighbor in V_0. The restrained Roman domination number of G, denoted by $\gamma_{rR}(G)$, is the minimum weight of an rRDF on G.

A function $f = (V_0, V_1, V_2)$ is called a $\gamma_{rR}(G)$-function if it is an rRDF on G and $f(V(G)) = \gamma_{rR}(G)$.

We present different bounds and characterizations for the restrained Roman domination number of a graph, and study restrained Roman domination in random graphs. In Section 2, we show that the restrained Roman domination problem is NP-complete for general graphs. In Section 3, we present different bounds and characterizations for the restrained Roman domination number. Finally in Section 4, we study restrained Roman domination in random graphs.

2 Complexity

In this section we prove that the restrained Roman domination decision problem is NP-complete. We shall prove the NP-completeness by reducing the following vertex cover decision problem, which is known to be NP-complete.

VERTEX COVER DECISION PROBLEM
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.
QUESTION: Does there exist a subset $C \subseteq V(G)$ of size at most k such that for each edge $xy \in E(G)$ we have $x \in C$ or $y \in C$?

Theorem 1 (Karp [14]) *Vertex cover decision problem is NP-complete for general graphs.*

RESTRAINED ROMAN DOMINATION PROBLEM
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.
QUESTION: Does there exist a restrained Roman dominating function for G with weight at most k?

Theorem 2 *The restrained Roman domination problem is NP-complete for general graphs.*

Proof. We transform the vertex cover decision problem for general graphs to the restrained Roman domination decision problem for general graphs. For a given graph $G = (V, E)$, let $s = 3|V(G)| + 4$ and construct a graph $H = (V_1, E_1)$ as follows. Let $V_1 = \{x_i \colon 1 \leq i \leq s\} \cup \{y\} \cup V(G) \cup \{e_i \colon e \in E, 1 \leq i \leq s\}$, and let

$$\begin{aligned} E_1 \quad = \quad & \{x_1 x_s\} \cup \{yv \colon v \in V\} \\ & \cup \{x_i x_{i+1} \colon 1 \leq i \leq s-1\} \\ & \cup \{x_i y \colon 1 \leq i \leq s\} \\ & \cup \{ve_i \colon v \in e,\, e \in E, 1 \leq i \leq s\} \\ & \cup \{e_i e_{i+1} (\text{mod } s) \colon v \in e,\, e \in E,\, 1 \leq i \leq s\}. \end{aligned}$$

Figure 1 shows the graph H obtained from $G = P_3$ by the above procedure. Note that

$$H[\{x_i\colon 1 \le i \le 13\}] \cong H[\{e_j\colon 1 \le j \le 13\}] \cong H[\{f_l\colon 1 \le l \le 13\}] \cong C_{13},$$

y is adjacent to x_i for $i = 1, 2, \ldots, 13$, e_j is adjacent to both a and b for $j = 1, 2, \ldots, 13$, and f_l is adjacent to both b and c for $l = 1, 2, \ldots, 13$.

We claim that G has a vertex cover of size at most k if and only if H has an rRDF with weight at most $2k+2$. Hence the NP-completeness of the restrained Roman domination problem in general graphs follows from that of the vertex cover problem. First, if G has a vertex cover C of size at most k, then the function f defined on V_1 by $f(v) = 2$ for $v \in C \cup \{y\}$ and $f(v) = 0$ otherwise, is an rRDF with weight at most $2k + 2$. On the other hand, suppose that H has an rRDF g with weight at most $2k + 2$. If $g(y) \neq 2$, then

$$\sum_{i=1}^{s} g(x_i) \ge \gamma_{rR}(C_s) \ge \gamma_R(C_s) \ge \frac{2s}{3} > 2|V(G)| + 2 \ge 2k + 2,$$

which is a contradiction. Thus $g(y) = 2$. Similarly, we have $g(u) = 2$ or $g(v) = 2$ for any $e = uv \in E$. Therefore $C = \{v \in V : g(v) = 2\}$ is a vertex cover of G and $2|C| + 2 \le w(g) \le 2k + 2$. Consequently, $|C| \le k$. ∎

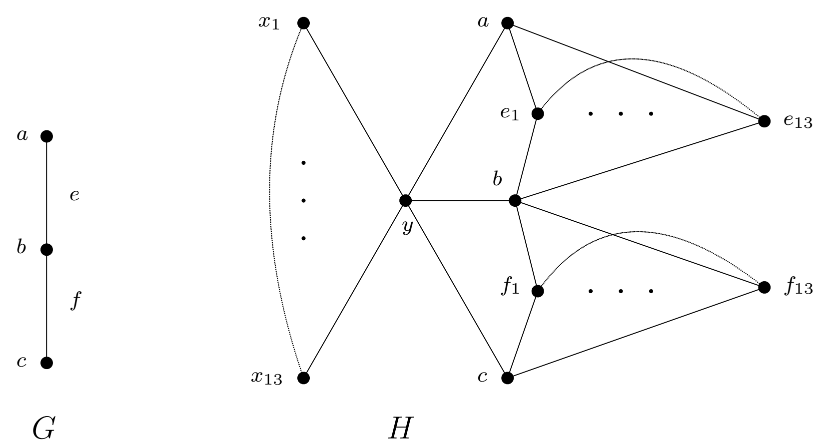

Figure 1: The graphs $G = P_3$ and H

3 Bounds on the restrained Roman domination number of a graph

In this section we present different bounds and characterizations concerning the restrained Roman domination number.

3.1 Trees

We prove that for every nontrivial tree T of diameter at least three, order n, with l leaves and s support vertices, we have $\gamma_{rR}(T) \geq (2n+l-s+4)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_4, P_5, P_6\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_3.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaves to a leaf of T_k adjacent to a weak support vertex.

We now prove a lower bound on the restrained Roman domination number of a tree. We also prove that for the equality to be satisfied, the tree must belong to the family $\mathcal{T}$.

Lemma 3 *For every tree T of diameter at least three, order n, with l leaves and s support vertices, we have $\gamma_{rR}(T) \geq (2n+l-s+4)/3$, and if the equality is satisfied, then $T \in \mathcal{T}$.*

Proof. First assume that $\text{diam}(T) = 3$. Thus T is a double star. We have $(2n+l-s+4)/3 = (2n+n-2-2+4)/3 = n = \gamma_{rR}(T)$. If $T = P_4$, then $T \in \mathcal{T}$. If T is a double star different from P_4, then it can be obtained P_4 by appropriate numbers of operations $\mathcal{O}_1$ performed on the support vertices. Thus $T \in \mathcal{T}$. Now assume that $\text{diam}(T) \geq 4$. Thus the order n of the tree T is at least five. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(T)$-function. First assume that some support vertex of T, say x, is strong. Let y and z be leaves adjacent to x. Let $T' = T - y$. We have $n' = n-1$, $l' = l-1$ and $s' = s$. Clearly, we have $y, z \in V_1 \cup V_2$. The function f is minimum, thus $f(y) \neq 2$ or $f(z) \neq 2$. Without loss of generality we assume that $f(y) = 1$. It is easy to observe that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 1$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1$ $\geq (2n'+l'-s'+4)/3 + 1 = (2n-2+l-1-s+7)/3 = (2n+l-s+4)/3$. If $\gamma_{rR}(T) = (2n+l-s+4)/3$, then obviously $\gamma_{rR}(T') = (2n'+l'-s'+4)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity diam(T). Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Clearly, $t \in V_1 \cup V_2$. Assume that $v \notin V_0$. Then $f(t) = 1$ as the function f is minimum. Let $T' = T - t$. We have $n' = n-1$, $l' = l$ and $s' \in \{s-1, s\}$. It is easy to observe that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 1$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1 \geq (2n' + l' - s' + 4)/3 + 1 \geq (2n - 2 + l - s + 7)/3 = (2n + l - s + 5)/3 > (2n + l - s + 4)/3$. Thus we can assume that $f(v) = 0$, and consequently, $f(u) = 0$ and $f(t) = 2$.

Assume that among the children of u there is a support vertex, say x, other than v. The leaf adjacent to x we denote by y. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Because of the similarity of the subtrees T_v and T_x, we can assume that $f(x) = 0$ and $f(y) = 2$. Observe that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 2$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 2 \geq (2n' + l' - s' + 4)/3 + 2 = (2n - 4 + l - 1 - s + 1 + 10)/3 = (2n + l - s + 6)/3 > (2n + l - s + 4)/3$.

Now assume that some child of u, say x, is a leaf. Clearly, $f(x) \neq 0$. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. If $f(x) = 1$, then obviously $f|_{V(T')}$ is an rRDF for the tree T'. Now assume that $f(x) = 2$. We have $f(w) \in \{0, 1\}$, otherwise we can change the value of $f(x)$ from 2 to 1, a contradiction to the minimality of f. If $f(w) = 1$, then let us observe that $f|_{V(T')}$ with the modification that $f(w) = 2$, is an rRDF for the tree T'. We now conclude that $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 1$. We get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1 \geq (2n' + l' - s' + 4)/3 + 1 = (2n - 2 + l - 1 - s + 1 + 7)/3 = (2n + l - s + 5)/3 > (2n + l - s + 4)/3$. Now assume that $f(w) = 0$. Let $T'' = T - T_v$. We have $n'' = n - 2$, $l'' = l - 1$ and $s'' = s - 1$. Observe that $f|_{V(T'')}$ is an rRDF for the tree T''. Therefore $\gamma_{rR}(T'') \leq \gamma_{rR}(T) - 2$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T'') + 2 \geq (2n'' + l'' - s'' + 4)/3 + 2 \geq (2n - 4 + l - 1 - s + 1 + 10)/3 = (2n + l - s + 6)/3 > (2n + l - s + 4)/3$.

Now assume that $d_T(u) = 2$. Thus $f(w) = 2$. First assume that there is a child of w other than u, say x, such that the distance of w to the most distant vertex of T_x is three. It suffices to consider only the possibility when T_x is a path P_3. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Observe that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 2$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 2 \geq (2n' + l' - s' + 4)/3 + 2 = (2n - 6 + l - 1 - s + 1 + 10)/3 = (2n + l - s + 4)/3$. If $\gamma_{rR}(T) = (2n + l - s + 4)/3$, then obviously $\gamma_{rR}(T') = (2n' + l' - s' + 4)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of w, say x, such that the distance of w to the most distant vertex of T_x is two. Thus x is a support vertex of degree two. The leaf adjacent to x we denote by y. Let $T' = T - T_x$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Let us observe that $f(x) = 1$ and $f(y) = 1$. It is easy to see that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 2$. We now get

$\gamma_{rR}(T) \geq \gamma_{rR}(T') + 2 \geq (2n' + l' - s' + 4)/3 + 2 = (2n - 4 + l - 1 - s + 1 + 10)/3 = (2n + l - s + 6)/3 > (2n + l - s + 4)/3$.

Now assume that some child of w, say x, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. Notice that $f(x) = 1$. Clearly, $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 1$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1 \geq (2n' + l' - s' + 4)/3 + 1 = (2n - 2 + l - 1 - s + 1 + 7)/3 = (2n + l - s + 5)/3 > (2n + l - s + 4)/3$.

Now assume that $d_T(w) = 2$. First assume that some child of d, say x, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. Notice that $f(x) = 1$. It is easy to see that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 1$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1 \geq (2n' + l' - s' + 4)/3 + 1 = (2n - 2 + l - 1 - s + 1 + 7)/3 = (2n + l - s + 5)/3 > (2n + l - s + 4)/3$.

Now assume that no child of d is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l$ and $s' = s$. If $\mathrm{diam}(T') \in \{1, 2\}$, then $T' \in \{P_2, P_3\}$ and $T \in \{P_5, P_6\} \subseteq \mathcal{T}$. Now assume that $\mathrm{diam}(T') \geq 3$. Observe that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 2$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 2 \geq (2n' + l' - s' + 4)/3 + 2 \geq (2n - 6 + l - s + 10)/3 = (2n + l - s + 4)/3$. If $\gamma_{rR}(T) = (2n + l - s + 4)/3$, then obviously $\gamma_{rR}(T') = (2n' + l' - s' + 4)/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$. ■

We have the following necessary condition for that a tree T satisfies the equality of the lower bound of $(2n + l - s + 4)/3$ on the restrained Roman domination number.

Lemma 4 *Let T be a tree. If $\gamma_{rR}(T) = (2n + l - s + 4)/3$, then for every leaf x of T, there is a $\gamma_{rR}(T)$-function f that assigns the value 2 to the vertex x.*

Proof. The neighbor of x we denote by y. Suppose that for every $\gamma_{rR}(T)$-function f we have $f(x) = 1$. Let $T' = T - x$. It is easy to observe that $f|_{V(T')}$ is an rRDF for the tree T'. Therefore $\gamma_{rR}(T') \leq \gamma_{rR}(T) - 1$. If y is a strong support vertex, then $n' = n - 1$, $l' = l - 1$ and $s' = s$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1 \geq (2n' + l' - s' + 4)/3 + 1 = (2n - 2 + l - 1 - s + 7)/3 = (2n + l - s + 4)/3$. After an appropriate number of analogical steps, we get a tree in which y is a weak support vertex. Thus we now assume that y is a weak support vertex. Then $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We now get $\gamma_{rR}(T) \geq \gamma_{rR}(T') + 1 \geq (2n' + l' - s' + 4)/3 + 1 = (2n - 2 + l - 1 - s + 1 + 7)/3 = (2n + l - s + 5)/3 > (2n + l - s + 4)/3$, a contradiction. ■

We now prove that for every tree T of the family $\mathcal{T}$ we have $\gamma_{rR}(T) = (2n + l - s + 4)/3$.

Lemma 5 *If $T \in \mathcal{T}$, then $\gamma_{rR}(T) = (2n + l - s + 4)/3$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T \in \{P_4, P_5, P_6\}$, then we have $(2n + l - s + 4)/3 = (2n + n - 2 - 2 + 4)/3 = n = \gamma_{rR}(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let n' be the order of the tree T', l' the number of its leaves and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n' + 1$, $l = l' + 1$ and $s = s'$. Let x be the attached vertex. Let $f' = (V_0, V_1, V_2)$ be a $\gamma_{rR}(T')$-function. It is easy to see that extending f' by $f'(x) = 1$ gives us an rRDF for the tree T. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 1$. We now get $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 1 = (2n' + l' - s' + 4)/3 + 1 = (2n - 2 + l - 1 - s + 7)/3 = (2n + l - s + 4)/3$. On the other hand, by Lemma 3 we have $\gamma_{rR}(T) \geq (2n + l - s + 4)/3$. This implies that $\gamma_{rR}(T) = (2n + l - s + 4)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n' + 3$, $l = l' + 1$ and $s = s' + 1$. The vertex to which is attached P_3 we denote by x. Let $v_1 v_2 v_3$ be the attached path. Let v_1 be joined to x. Let abc denote a path P_3 adjacent to x and different from $v_1 v_2 v_3$. Let a and x be adjacent. By Lemma 4, there is a $\gamma_{rR}(T')$-function f' such that $f'(c) = 2$. Consequently, $f'(b) = 0$, $f'(a) = 0$ and $f'(x) = 2$. It is easy to observe that extending f' by letting v_1 and v_2 have the weight 0 and v_3 the weight 2, we get an rRDF for the tree T. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 2$. We now get $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 2 = (2n' + l' - s' + 4)/3 + 2 = (2n - 6 + l - 1 - s + 1 + 10)/3 = (2n + l - s + 4)/3$. This implies that $\gamma_{rR}(T) = (2n + l - s + 4)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n' + 3$, $l = l'$ and $s = s'$. The leaf to which is attached P_3 we denote by x. Let $v_1 v_2 v_3$ be the attached path. Let v_1 be joined to x. By Lemma 4, there is a $\gamma_{rR}(T')$-function f' such that $f'(x) = 2$. It is easy to observe that extending f' by $f'(v_1) = 0$, $f'(v_2) = 0$ and $f'(v_3) = 2$, we get an rRDF for the tree T. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 2$. We now get $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 2 = (2n' + l' - s' + 4)/3 + 2 = (2n - 6 + l - s + 10)/3 = (2n + l - s + 4)/3$. This implies that $\gamma_{rR}(T) = (2n + l - s + 4)/3$. ■

As an immediate consequence of Lemmas 3 and 5, we get the following lower bound on the restrained Roman domination number of a tree together with the characterization of the extremal trees.

Theorem 6 *For every tree T of diameter at least three, order n, with l leaves and s support vertices, we have $\gamma_{rR}(T) \geq (2n + l - s + 4)/3$, with equality if and only if $T \in \mathcal{T}$.*

3.2 General graphs

Pushpam et al. [17] determined the restrained Roman domination number of paths and cycles by considering different cases of n modulo 3. We first show the following property of graphs in which some vertex is neither a leaf nor a support vertex.

Proposition 7 *If a graph G has an edge non-incident with a leaf, then there is a $\gamma_{rR}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_0 \neq \emptyset$.*

Proof. The result is obvious if $\gamma_{rR}(G) < n$. Thus assume that $\gamma_{rR}(G) = n$. Let xy be an edge of G such that $d_G(x) \geq 2$ and $d_G(y) \geq 2$. If $N_G(x) \cap N_G(y) \neq \emptyset$, then let a be a common neighbor of x and y. Observe that $(\{x, y\}, V(G) \setminus \{x, y, a\}, \{a\})$ is an rRDF for G of weight $n-1$, a contradiction. Now assume that $N_G(x) \cap N_G(y) = \emptyset$. Let $x_1 \in N_G(x) \setminus N_G(y)$ and $y_1 \in N_G(y) \setminus N_G(x)$. Then $(\{x, y\}, V(G) \setminus \{x, x_1, y, y_1\}, \{x_1, y_1\})$ is a desired $\gamma_{rR}(G)$-function. ■

For any odd integer $n \geq 3$, let G_n be the graph obtained from $(n-1)/2$ copies of K_2 by adding a new vertex and joining it to both leaves of each K_2. Thus $G_3 = K_3$, and for $n > 3$, G_n has a vertex of degree $n-1$ and all other vertices of degree two. Figure 2 shows the graph G_7. Let $\mathcal{G} = \{G_n \colon n \geq 3 \text{ is odd}\}$.

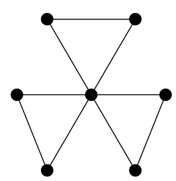

Figure 2: The graph G_7

Theorem 8 *For every connected graph G of order $n \geq 3$ with m edges we have $\gamma_{rR}(G) \geq n + 1 - 2m/3$, with equality if and only if $G \in \mathcal{G}$.*

Proof. If G is a star, then $\gamma_{rR}(G) = n > (n+5)/3 = n+1-2m/3$. Thus assume that G is not a star. By Proposition 7, there is a $\gamma_{rR}(G)$-function $f = (V_0, V_1, V_2)$ such that $V_0 \neq \emptyset$. Clearly, $G[V_0]$ has no isolated vertex. Thus $|E(G[V_0])| \geq |V_0|/2$. Let $E(V_0, V_2)$ be the set of edges between vertices of V_0 and vertices of V_2. Since every vertex of V_0 has a neighbor in V_2, we have $|E(V_0, V_2)| \geq |V_0|$. We have

$$m \geq |E(G[V_0])| + |E(V_0, V_2)| \geq \frac{1}{2}|V_0| + |V_0| = \frac{3}{2}|V_0|, \qquad (1)$$

and thus $|V_0| \leq 2m/3$. We now get

$$n = |V_0| + |V_1| + |V_2| \leq \frac{2}{3}m + \gamma_{rR}(G) - |V_2| \leq \frac{2}{3}m + \gamma_{rR}(G) - 1. \qquad (2)$$

This implies that $\gamma_{rR}(G) \geq n + 1 - 2m/3$.

It is easy to see that for any odd integer $n \geq 3$ we have $\gamma_{rR}(G_n) = 2$ and $|E(G_n)| = 3(n-1)/2$. Consequently, $\gamma_{rR}(G) = n + 1 - 2m/3$. Now assume that G is a graph with $\gamma_{rR}(G) = n + 1 - 2m/3$. Then all inequalities of (1) and (2) become equalities. We have $|V_2| = 1$. Let $V_2 = \{x\}$. Since $|E(G[V_0])| = |V_0|/2$, the set V_0 has an even number of vertices, and each component of $G[V_0]$ is a K_2. From $|E(V_0, V_2)| = |V_0|$ we obtain that every vertex of V_0 is adjacent to x. Since G is connected, we have $|V_1| = \emptyset$. Consequently, $G \in \mathcal{G}$. ■

It is proved in [17] that for a path P_n and a cycle C_n we have $\gamma_{rR}(P_n) \leq (2n+6)/3$ and $2n/3 \leq \gamma_{rR}(C_n) \leq (2n+5)/3$. We have the following upper bound on the restrained Roman domination number of a graph in terms of its diameter.

Proposition 9 *For every connected graph G of order n we have $\gamma_{rR}(G) \leq n + 1 - \lfloor (diam(G) - 2)/3 \rfloor$, and this bound is sharp.*

Proof. Let $P = v_0 v_1 \ldots v_{\mathrm{diam}(G)}$ be a diametrical path in G, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(P)$-function. Then $w(f) \leq (2(\mathrm{diam}(G) + 1) + 6)/3$. Define g on $V(G)$ by letting $g(x) = f(x)$ for $x \in V(P)$, while $g(x) = 1$ if $x \in V(G) \setminus V(P)$. Obviously, g is an rRDF for G. Hence $\gamma_{rR}(G) \leq w(f) + n - \mathrm{diam}(G) - 1 \leq (2(\mathrm{diam}(G) + 1) + 6)/3 + n - \mathrm{diam}(G) - 1 = n + 1 - (\mathrm{diam}(G) - 2)/3 \leq n + 1 - \lfloor (\mathrm{diam}(G) - 2)/3 \rfloor$. To see the sharpness, consider the path P_6. ■

Similarly we obtain the following bound in terms of the girth and the order.

Proposition 10 *For every connected graph G of order n and girth $g(G)$ we have $\gamma_{rR}(G) \leq n + 1 - \lfloor (g(G) - 2)/3 \rfloor$, and this bound is sharp.*

We have the following upper bound on the restrained Roman domination number of a graph in terms of its order and size.

Proposition 11 *If G is a connected graph of order n with m edges, then $\gamma_{rR}(G) \leq 2m - n + 2$, with equality if and only if G is a tree with $\gamma_{rR}(G) = n$.*

Proof. Since G is connected, we have $m \geq n - 1$. We now get $\gamma_{rR}(G) \leq n = 2(n-1) - n + 2 \leq 2m - n + 2$. If $\gamma_{rR}(G) = 2m - n + 2$, then $m = n - 1$ and thus G is a tree with $\gamma_{rR}(G) = n$. The converse is obvious. ■

3.3 Probabilistic bounds

For the probabilistic methods we follow [1]. Analogously to some results of [22], we obtain several probabilistic bounds for the restrained Roman domination number. Cockayne et al. [3] presented the following bound for the Roman domination number of a graph.

Theorem 12 (Cockayne et al. [3]) *Let G be a graph of order n. If $\delta > 0$, then*

$$\gamma_R(G) \leq n\left(\frac{2\ln(1+\delta) - \ln 4 + 2}{\delta + 1}\right).$$

We show that the above bound holds for the restrained Roman domination if $n < \delta(\delta - 1)/(\ln \delta - \ln 2 + 1)$.

Theorem 13 *Let G be a graph of order n. If $\delta > 0$ and $n < \delta(\delta - 1)/(\ln \delta - \ln 2 + 1)$, then*

$$\gamma_{rR}(G) \leq n\left(\frac{2\ln(1+\delta) - \ln 4 + 2}{\delta + 1}\right).$$

Proof. The condition $n < \delta(\delta-1)/(\ln \delta - \ln 2 + 1)$ implies that $\delta > n \cdot (\ln \delta - \ln 2 + 1)/\delta + 1$. By Theorem 12 we have $\gamma_R(G) \leq n \cdot (2\ln(1+\delta) - \ln 4 + 2)/(\delta + 1)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function with $w(f) \leq n \cdot (2\ln(1+\delta) - \ln 4 + 2)/(\delta+1)$ such that $|V_1|$ is minimum. Then any vertex of V_0 is adjacent to at most one vertex of V_1. Obviously, $|V_2| \leq \gamma_R(G)/2$. Let $v \in V_0$. Then $d_G(v) \geq \delta > n \cdot (\ln \delta - \ln 2 + 1)/\delta + 1 > n \cdot (\ln(\delta + 1) - \ln 2 + 1)/\delta + 1 \geq \gamma_R(G)/2 + 1 \geq |V_2| + 1$. By the choice of f, the vertex v is adjacent to some vertex in V_0. Thus f is an rRDF and we get $\gamma_{rR}(G) \leq w(f) \leq n(2\ln(1+\delta) - \ln 4 + 2)/(\delta + 1)$. ■

We have the following upper bound on the restrained Roman domination number of a graph with no leaves and cut vertices.

Theorem 14 *If G is a chordal graph of order n with no cut vertex and $\delta(G) \geq 2$, then*

$$\gamma_{rR}(G) \leq 2n\left(1 - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}}\right).$$

Proof. Given a chordal graph G with no cut vertex and with minimum degree at least two, we select a set of vertices A, where each vertex is selected independently with probability $p = 1 - (1/(\delta + 1))^{1/\delta}$. Let $B = V \setminus N[A]$ be the set of vertices not dominated by A. Let $C \subseteq V(G) \setminus (A \cup B)$ be the set of all vertices c such that $N_G(c) \subseteq A \cup B$. It follows from the assumption that for every $c \in C$ we have $N_G(c) \subseteq A$. Then $f = (V(G) \setminus (A \cup B \cup C), B \cup C, A)$ is an rRDF. Clearly, $Pr(x \in A) = p$, $Pr(x \in B) = (1-p)^{1+d_G(x)} \leq (1-p)^{1+\delta}$ and $Pr(x \in C) = (1-p) \cdot p^{d_G(x)} \leq (1-p) \cdot p^{\delta}$. Since $\delta \geq 2$, we have $p < 1/2$. We now bound the expected weight of f,

$$\begin{aligned} E(w(f)) &\leq np + n(1-p)^{1+\delta} + n(1-p)p^{\delta} && (3) \\ &\leq 2n(p + (1-p)^{1+\delta}) && (4) \\ &\leq 2n\left(1 - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}}\right). && (5) \end{aligned}$$

Now the result follows from the Pigeonhole property of expectation. ■

Corollary 15 *Let G be a chordal graph of order n with no cut vertex. If $\delta(G) \geq 2$ and $\ln(1+\delta)/(1+\delta) \leq 1/2$, then*

$$\gamma_{rR}(G) \leq 2n\left(\frac{\ln(1+\delta)+1}{\delta+1}\right).$$

Proof. We select a set of vertices A independently and randomly with probability $p = \ln(1+\delta)/(1+\delta)$ and follow the proof of Theorem 14. Using the inequality $e^{-x} \geq 1-x$ for each $x \geq 0$ we obtain from (3) that

$$\begin{aligned} E(w(f)) &\leq 2n(p+(1-p)^{1+\delta}) \\ &\leq 2n(p+e^{-p(1+\delta)}) \\ &\leq 2n\left(\frac{\ln(1+\delta)+1}{\delta+1}\right). \end{aligned}$$

■

Let $\beta(G)$ denote the maximum size of a matching in G.

Theorem 16 *For every graph G of order n we have*

$$\gamma_{rR}(G) \leq \frac{3\ln(\delta+1)+\delta+4-\ln 8}{\delta+1} \cdot n - 2\beta(G).$$

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function such that $|V_2|$ is minimum. Then each vertex of V_2 has a private neighbor in V_0. Let M be a maximum matching of G. Without loss of generality we may assume that $M = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_{\beta(G)}\}$, where e_i has at least one end-point in V_2, for $i = 1, 2, \ldots, k$. Thus $k \leq |V_2|$ and G has at least $\beta(G)-k \geq \beta(G)-|V_2|$ edges with both end-points in $V(G)\setminus V_2$. Let $e_{i1}, e_{i2}, \ldots, e_{it}$ be the edges of M which have exactly one end-point in V_2. Then

$$g = (V_0\setminus\{x\colon x \in e_j, j = k+1, \ldots, \beta(G)\}, V_1\cup\{x\colon x \in e_j, j = k+1, \ldots, \beta(G)\}, V_2)$$

is an rRDF for G. We now get

$$\begin{aligned} \gamma_{rR}(G) &\leq |V_1| + 2|V_2| + t \\ &\leq |V_1| + 2|V_2| + |V_0| - 2(\beta(G) - k) \\ &= |V_1| + 2|V_2| + |V_0| - 2\beta(G) + 2k \\ &\leq |V_1| + 2|V_2| + |V_0| - 2\beta(G) + 2|V_2| \\ &= |V_1| + 4|V_2| + (n - |V_1| - |V_2|) - 2\beta(G) \\ &= n + 3|V_2| - 2\beta(G). \end{aligned}$$

By Theorem 12 we have $|V_2| \leq \gamma_R(G)/2 \leq n \cdot (2\ln(1+\delta) - \ln 4 + 2)/2(\delta+1)$. Thus

$$\begin{aligned} \gamma_{rR}(G) &\leq n + \frac{3n}{2}\left(\frac{2\ln(1+\delta) - \ln 4 + 2}{\delta+1}\right) - 2\beta(G) \\ &= \frac{3\ln(\delta+1) + \delta + 4 - \ln 8}{\delta+1} \cdot n - 2\beta(G). \end{aligned}$$

■

Corollary 17 *If a graph G of order n has a perfect matching, then*

$$\gamma_{rR}(G) \leq \frac{3\ln(\delta+1) + 3 - \ln 8}{\delta+1} \cdot n.$$

4 Restrained Roman domination in random graphs

Let n be a positive integer and let $p \in \{0,1\}$. The random graph $G(n,p)$ is a probability space over the set of graphs on the vertex set $[n] = \{1,\ldots,n\}$ determined by $Pr[\{i,j\} \in E(G)] = p$ with these events mutually independent (see for example [2]). We say that a random graph $G = G(n,p)$ satisfies a property Q if the probability that G has the property Q tends to 1 as n tends to infinity. We also say that almost all graphs satisfy the property Q if a random graph satisfies Q.

Theorem 18 (Weber [20]) *For almost every graph $G(n,p)$ we have $k+1 \leq \gamma(G) \leq k+2$, where $k = \lfloor \log n - 2\log\log n + \log\log e \rfloor$ and the logarithm is in base $1/(p-1)$.*

It follows from the above theorem that $\gamma(G) = \log n(1+o(1))$. We show that the restrained Roman domination number is equal to the Roman domination number in a random graph.

Theorem 19 *For almost every graph G we have $\gamma_{rR}(G) = \gamma_R(G)$.*

Proof. From Theorem 12 of [2] we know that for a random graph $G = G(n,p)$ we have

$$\left|\delta - pn - (2pqn\log n)^{\frac{1}{2}} + \left(\frac{pqn}{8\log n}\right)^{\frac{1}{2}} \log\log\log n\right| \leq c(n)\left(\frac{n}{\log n}\right)^{\frac{1}{2}},$$

where $c(n)$ tends to infinity arbitrarily slowly, and the logarithm is in base $1/(p-1)$. Thus we obtain $\delta = pn(1+o(1))$. Let D be a $\gamma(G)$-set. Thus $|D| = \log n(1+o(1))$. It is obvious that $f = (V(G) \setminus D, \emptyset, D)$ is an RDF for G. We have $\gamma_R(G) = w(f) \leq 2|D| = 2\log n(1+o(1))$. Let $g = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function such that V_1 is minimum. Then each vertex of V_0 is adjacent to at most one vertex of V_1. Let $v \in V_0$. Then $d_G(v) \geq \delta = pn(1 + o(1))$. If n is sufficiently large, then $pn(1 + o(1)) > 2 \log n(1 + o(1)) + 1 \geq |V_2| + 1$. Thus v is adjacent to some vertex in V_0. Consequently, g is an rRDF. Thus $\gamma_{rR}(G) \leq \gamma_R(G)$ and the result follows. ■

References

[1] N. Alon and J. Spencer, *The Probabilistic Method,* John Wiley, New York, (1992).

[2] B. Bollobás, *Random Graphs,* Cambridge University Press (2001).

[3] E. Cockayne, P. Dreyer, S. Hedetniemi and S. Hedetniemi, *Roman domination in graphs,* Discrete Mathematics 278 (2004), 11–22.

[4] P. Dankelmann, J. Hattingh, M. Henning and H. Swart, *Trees with equal domination and restrained domination numbers,* Journal of Global Optimization 34 (2006), 597–607.

[5] G. Domke, J. Hattingh, S. Hedetniemi, R. Laskar and L. Markus, *Restrained domination in graphs,* Discrete Mathematics 203 (1999), 61–69.

[6] G. Domke, J. Hattingh, M. Henning and L. Markus, *Restrained domination in trees,* Discrete Mathematics 211 (2000), 1–9.

[7] G. Domke, J. Hattingh, M. Henning and L. Markus, *Restrained domination in graphs with minimum degree two,* Journal of Combinatorial Mathematics and Combinatorial Computing 35 (2000), 239–254.

[8] S. Fujita and M. Furuya, *Difference between 2-rainbow domination and Roman domination in graphs,* Discrete Applied Mathematics 161 (2013), 806–812.

[9] J. Hattingh and M. Henning, *Characterizations of trees with equal domination parameters,* Journal of Graph Theory 34 (2000), 142–153.

[10] J. Hattingh and E. Joubert, *Restrained domination in cubic graphs,* Journal of Combinatorial Optimization 22 (2011), 166–179.

[11] T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

[12] M. Henning, *Graphs with large restrained domination number,* 16th British Combinatorial Conference (London, 1997). Discrete Mathematics 197/198 (1999), 415–429.

[13] K. Kämmerling and L. Volkmann, *Roman k-domination in graphs,* Journal of the Korean Mathematical Society 46 (2009), 1309–1318.

[14] R. Karp, *Reducibility among combinatorial problems,* in: Complexity of Computer Computations (Proc. Sympos., IBM Thomas J. Watson Research Center, Yorktown Heights, New York, 1972), 1972, 85–103.

[15] C-H. Liu and G. Chang, *Roman domination on strongly chordal graphs,* Journal of Combinatorial Optimization 26 (2013), 608–619.

[16] C. ReVelle and K. Rosing, *Defendens imperium romanum: a classical problem in military strategy,* American Mathematical Monthly 107 (2000), 585–594.

[17] P. Pushpam and S. Padmapriea, *Restrained Roman Domination in Graphs,* Transactions on Combinatorics (2013).

[18] I. Stewart, *Defend the Roman Empire!,* Scientific American 281 (1999), 136–139.

[19] J. Telle and A. Proskurowski, *Algorithms for vertex partitioning problems on partial k-trees,* SIAM Journal on Discrete Mathematics 10 (1997), 529–550.

[20] K. Weber, *Domination number for almost every graph,* Rostocker Mathematisches Kolloquium 16 (1981), 31–43.

[21] H. Xing, X. Chen and X. Chen, *A note on Roman domination in graphs,* Discrete Mathematics 306 (2006), 3338–3340.

[22] V. Zverovich and A. Poghosyan, *On Roman, Global and Restrained Domination in Graphs,* Graphs and Combinatorics 27 (2011), 755–768.