On interval and indifference graphs

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Abstract

Skrien characterized all graphs whose line graphs are interval graphs and indifference graphs. We determine all graphs whose middle graphs (total graphs, respectively) are interval graphs and indifference graphs.
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1 Introduction

By a graph we mean a simple graph $G = (V, E)$ with vertex set $V(G) = V$ and edge set $E(G) = E$. If v is a vertex of a graph G, then $N_G(v)$ denotes the neighborhood of v in G, that is, the set of vertices adjacent to v. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. Three vertices v_1, v_2, v_3 in G are said to form an asteroidal triple $[v_1, v_2, v_3]$ if v_i and v_k are in the same connected component of $G - N_G(v_j)$. In other words, any two of them are connected by a path in G which avoids the neighborhood of the remaining vertex.

A graph G is called an interval graph if it is possible to assign to each vertex of G a closed interval on the real line such that two distinct vertices of G are adjacent if and only if the corresponding intervals have a non-empty intersection, that is, if there exists a collection $\mathcal{J} = \{I_v : v \in V(G)\}$ of closed intervals on the real line such that G is isomorphic to the intersection graph $\Omega(\mathcal{J})$ of $\mathcal{J}$. The first characterization of interval graphs was given by Lekkerkerker and Boland in [3].

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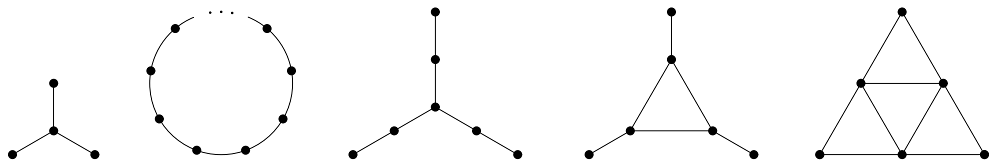
Theorem 1 (Lekkerkerker and Boland) *A graph G is an interval graph if and only if it does not contain C_k $(k \geq 4)$ as an induced subgraph, and also no asteroidal triple.*

A graph G is called an indifference graph if there exists a real-valued function f on $V(G)$ and a real number $\delta > 0$ such that $vu \in E(G)$ if and only if $|f(u) - f(v)| \leq \delta$ for $u, v \in V(G)$. Fundamental structural properties of the indifference graphs were proved by Roberts in [4].

Theorem 2 (Roberts) *A graph G is an indifference graph if and only if it does not contain any of the graphs $K_{1,3}$, C_k $(k \geq 4)$, G_2, G_3 (see Figure 1) as an induced subgraph.*



$K_{1,3}$ $\quad$ C_k $(k \geq 4)$ $\quad$ G_1 $\quad$ G_2 $\quad$ G_3

Figure 1

The interval and indifference graphs clearly satisfy the hereditary property, that is, every induced subgraph is also an interval or indifference graph, respectively.

Several authors noticed that interval graphs and indifference graphs arise naturally in many contexts, and are useful, for example, in conflicting events, scheduling and allocation problems. For algorithmic aspects and various applications of interval and indifference graphs, see [2].

Skrien characterized all graphs whose line graphs are interval graphs and indifference graphs. We determine all graphs whose middle graphs (total graphs, respectively) are interval graphs and indifference graphs.

2 Interval and indifference line graphs

The line graph of a graph G, denoted by $L(G)$, is the intersection graph $\Omega(\overline{E}(G))$ of the family $\overline{E}(G) = \{\{u, v\} \colon uv \in E(G)\}$, that is, $L(G)$ is the graph whose vertices are in one-to-one correspondence with the edges of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. Skrien [5] characterized the interval and indifference line graphs.

Theorem 3 (Skrien) *For a connected graph G the following conditions are equivalent:*

(*a*) *$L(G)$ is an interval graph;*

(*b*) *$L(G)$ is an indifference graph;*

(*c*) *$L(G)$ does not contain any of the graphs G_2, G_3, C_k $(k \geq 4)$ (see Figure 1) as an induced subgraph;*

(*d*) *G does not contain any of the graphs G_1, G_2, C_k $(k \geq 4)$ (see Figure 1) as a subgraph;*

(*e*) *G is a subgraph of the graph T_m^d $(d \geq 2,\ m \geq 0)$ given in Figure 2.*

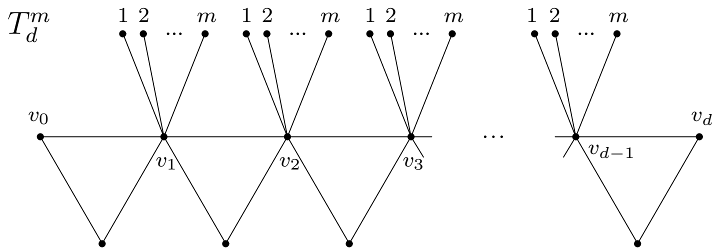


Figure 2

3 Interval and indifference middle graphs

The middle graph $M(G)$ of a graph G is defined to be the intersection graph $\Omega(F)$ of the family $F = \overline{V}(G) \cup \overline{E}(G) = \{\{v\} \colon v \in V(G)\} \cup \{\{u,v\} \colon uv \in E(G)\}$. In [2] it is proved that $M(G)$ is isomorphic to the line graph $L(G \circ K_1)$ of the corona $G \circ K_1$ of G and K_1. (The graph $G \circ K_1$ is a graph obtained by taking G and $|V(G)|$ copies of K_1, and joining the i-th vertex of G to the i-th copy of K_1.)

The next characterization of interval graphs immediately follows from Theorems 2 and 3 and from the equation $M(G) = L(G \circ K_1)$.

Corollary 4 *Let G be a connected graph. Then $M(G)$ is an interval graph if and only if G is a path.*

We now characterize graphs G for which the middle graph $M(G)$ is an indifference graph.

Theorem 5 *Let G be a connected graph. Then the middle graph $M(G)$ is an indifference graph if and only if G is a path.*

Proof. If G is a path on n vertices, then $G \circ K_1$ is a subgraph of T_d^m (given in Figure 2) for $d \geq n+1$ and $m \geq 1$. In the light of Theorem 3 and the equation $M(G) = L(G \circ K_1)$, the middle graph $M(G)$ is an indifference graph.

If $M(G) = L(G \circ K_1)$ is an indifference graph, then by Theorem 3, the graph $G \circ K_1$ cannot contain any of the graphs G_1, $G_2 = K_3 \circ K_1$, C_k ($k \geq 4$) as a subgraph. This implies that G does not contain any of the graphs $K_{1,3}$ and C_k ($k \geq 4$) as a subgraph. Thus G must be a path. ■

4 Interval and indifference total graphs

The total graph of a graph G, denoted by $T(G)$, is the intersection graph $\Omega(F)$ of the family $F = \overline{E}(G) \cup \overline{VE}(G) = \{\{u, v\} \colon uv \in E(G)\} \cup \{\{u\} \cup \{\{u, v\} \colon v \in N_G(u)\} \colon u \in V(G)\}$, that is, $T(G)$ is the graph for which there exists a one-to-one correspondence between its vertices and the vertices and edges of G such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements in G are adjacent or incident. This concept was originated by Behzad [1]. It is interesting to note that the graphs G and $L(G)$ are induced subgraphs of the total graph $T(G)$.

In the following theorem we characterize graphs whose total graphs are interval and indifference graphs.

Theorem 6 *For a connected graph G the following conditions are equivalent:*

(a) $T(G)$ *is an interval graph;*

(b) $T(G)$ *is an indifference graph;*

(c) G *is a path.*

Proof. $(a) \Rightarrow (c)$. Assume that $T(G)$ is an interval graph. Then its subgraph $L(G)$ is an interval graph, and by Theorem 3, the graph G is a subgraph of T_d^m for some d and m. The same arguments as above show that C_3 cannot be a subgraph of G, and therefore G is a tree. We now claim that the degree of each vertex v in G is at most two. Suppose to the contrary that v is a vertex of degree at least three in G, and let v_1, v_2 and v_3 be three neighbors of v in G. Then $T(G)$ contains an asteroidal triple $[v_1, v_2, v_3]$ isomorphic to G_2, a contradiction (see Theorem 1). Hence G is a path.

$(b) \Rightarrow (c)$. If the total graph $T(G)$ is an indifference graph, then its induced subgraphs G and $L(G)$ are also indifference graphs. In the light of Theorem 2, the graph G cannot contain $K_{1,3}$ as an induced subgraph. Furthermore, by Theorem 3, the graph G is a subgraph of T_d^m (given in Figure 2) for some d and m. This implies that G is a subgraph of T_d^0. Now let us observe that if G contains a cycle $C_3 = (v_0, v_1, v_2, v_0)$ of length 3, then $T(G)$ has an induced cycle

$C_4 = (v_0, v_1, v_1v_2, v_2v_0, v_0)$ of length 4 and $T(G)$ cannot be an indifference graph (see Theorem 2). From this it follows that G must be a path.

$(c) \Rightarrow (a)$ and (b). If G is a path, $G = P_n$, then it is easy to verify that the total graph $T(P_n)$ (see Figure 3 for $n = 4$) does not contain $K_{1,3}$, G_2, G_3, C_k $(k \geq 4)$ as an induced subgraph, and also no asteroidal triple. According to Theorems 1 and 2, the total graph $T(P_n)$ is an interval and indifference graph, see Figure 3 for an interval representation of $T(P_4)$. ∎

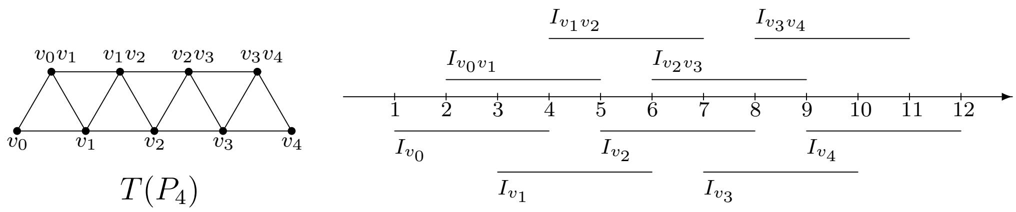


Figure 3

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