Minimal 2-dominating sets in trees

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Abstract

We provide an algorithm for listing all minimal 2-dominating sets of a tree of order n in time $\mathcal{O}(1.3248^n)$. This implies that every tree has at most 1.3248^n minimal 2-dominating sets. We also show that this bound is tight.

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1 Introduction

Let $G = (V, E)$ be a graph. The order of a graph is the number of its vertices. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity

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among all vertices of G. By P_n we denote a path on n vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. A dominating (2-dominating, respectively) set D is minimal if no proper subset of D is a dominating (2-dominating, respectively) set of G. A minimal 2-dominating set is abbreviated as m2ds. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [7], and further studied for example in [2, 10, 17]. For a comprehensive survey of domination in graphs, see [11, 12].

Observation 1 *Every leaf of a graph G is in every 2-dominating set of G.*

One of the typical questions in graph theory is how many subgraphs of a given property can a graph on n vertices have. For example, the famous Moon and Moser theorem [16] says that every graph on n vertices has at most $3^{n/3}$ maximal independent sets.

Combinatorial bounds are of interest not only on their own, but also because they are used for algorithm design as well. Lawler [15] used the Moon-Moser bound on the number of maximal independent sets to construct an $(1 + \sqrt[3]{3})^n \cdot n^{\mathcal{O}(1)}$ time graph coloring algorithm, which was the fastest one known for twenty-five years. In 2003 Eppstein [6] reduced the running time of a graph coloring to $\mathcal{O}(2.4151^n)$. In 2006 the running time was reduced [1, 14] to $\mathcal{O}(2^n)$. For an overview of the field, see [9].

Fomin et al. [8] constructed an algorithm for listing all minimal dominating sets of a graph on n vertices in time $\mathcal{O}(1.7159^n)$. There were also given graphs ($n/6$ disjoint copies of the octahedron) having $15^{n/6} \approx 1.5704^n$ minimal dominating sets. This establishes a lower bound on the running time of an algorithm for listing all minimal dominating sets of a given graph.

The number of maximal independent sets in trees was investigated in [18]. Couturier et al. [5] considered minimal dominating sets in various classes of graphs. The authors of [13] investigated the enumeration of minimal dominating sets in graphs.

Bród and Skupień [3] gave bounds on the number of dominating sets of a tree.

They also characterized the extremal trees. The authors of [4] investigated the number of minimal dominating sets in trees containing all leaves.

We provide an algorithm for listing all minimal 2-dominating sets of a tree of order n in time $\mathcal{O}(1.3248^n)$. This implies that every tree has at most 1.3248^n minimal 2-dominating sets. We also show that this bound is tight.

2 Listing algorithm

In this section we describe a recursive algorithm which lists all minimal 2-dominating sets of a given input tree T. The iterator of the solutions is denoted by $\mathcal{F}(T)$.

Algorithm

Notice that the diameter of a tree can be easily determined in a polynomial time.

Let T be a tree. If $\operatorname{diam}(T) = 0$, then $T = P_1 = v_1$. Let $\mathcal{F}(T) = \{\{v_1\}\}$. If $\operatorname{diam}(T) = 1$, then $T = P_2 = v_1v_2$. Let $\mathcal{F}(T) = \{\{v_1, v_2\}\}$. If $\operatorname{diam}(T) = 2$, then T is a star. By x we denote the support vertex of T. Let $\mathcal{F}(T) = \{V(T) \setminus \{x\}\}$.

Now consider trees T with $\operatorname{diam}(T) \geq 3$. Thus the order n of the tree T is at least four.

If some support vertex of T, say x, is adjacent to at least three leaves (we denote one of them by y), then let $T' = T - y$ and

$$\mathcal{F}(T) = \{D' \cup \{y\} \colon D' \in \mathcal{F}(T')\}.$$

Now consider trees T, in which every support vertex is adjacent to at most two leaves. The tree T can easily be rooted at a vertex r of maximum eccentricity $\operatorname{diam}(T)$ in polynomial time. A leaf, say t, at maximum distance from r, can also be easily computed in polynomial time. Let v denote the parent of t and let u denote the parent of v in the rooted tree. If $\operatorname{diam}(T) \geq 4$, then let w denote the parent of u. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

If $d_T(v) = 3$, then by a we denote the leaf adjacent to v and different from t. Let $T' = T - T_v$ and $T'' = T - t - a$, and let $\mathcal{F}(T)$ be as follows,

$$\begin{aligned} & \{D' \cup \{t, a\} \colon D' \in \mathcal{F}(T')\} \\ \cup \quad & \{D'' \cup \{t, a\} \colon D'' \in \mathcal{F}(T'') \text{ and } D'' \setminus \{v\} \notin \mathcal{F}(T')\}. \end{aligned}$$

If $d_T(v) = 2$ and $d_T(u) \geq 3$, then let $T' = T - T_v$, $T'' = T - T_u$, and

$$\mathcal{F}(T) = \{D' \cup \{t\} \colon u \in D' \in \mathcal{F}(T')\} \cup \{D'' \cup V(T_u) \setminus \{u\} \colon D'' \in \mathcal{F}(T'')\}.$$

If $d_T(v) = d_T(u) = 2$, then let $T' = T - T_v$, $T'' = T - T_u$, and

$$\mathcal{F}(T) = \{D' \cup \{t\} \colon D' \in \mathcal{F}(T')\} \cup \{D'' \cup \{v, t\} \colon w \in D'' \in \mathcal{F}(T'')\}.$$

3 Bounding the number of minimal 2-dominating sets

Now we prove that the running time of the algorithm from the previous section is $\mathcal{O}(1.3248^n)$.

Theorem 2 *For every tree T of order n, the algorithm from the previous section lists all minimal 2-dominating sets in time $\mathcal{O}(1.3248^n)$.*

Proof. We prove that the running time of the algorithm is $\mathcal{O}(1.3248^n)$. Moreover, we prove that the number of minimal 2-dominating sets of T is at most α^n, where $\alpha \approx 1.3248$ is the positive solution of the equation $x^3 - x - 1 = 0$.

We proceed by induction on the number n of vertices of a tree T. If $\mathrm{diam}(T) = 0$, then $T = P_1 = v_1$. Obviously, $\{v_1\}$ is the only m2ds of the path P_1. We have $n = 1$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha$. If $\mathrm{diam}(T) = 1$, then $T = P_2 = v_1v_2$. It is easy to see that $\{v_1, v_2\}$ is the only m2ds of the path P_2. We have $n = 2$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha^2$. If $\mathrm{diam}(T) = 2$, then T is a star. By x we denote the support vertex of T. It is easy to observe that $V(T) \setminus \{x\}$ is the only m2ds of the tree T. We have $n \geq 3$ and $|\mathcal{F}(T)| = 1$. Obviously, $1 < \alpha^n$.

Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. The results we obtain by the induction on the number n. Assume that they are true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is adjacent to at least three leaves. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a m2ds of the tree T'. Obviously, $D' \cup \{y\}$ is an m2ds of T. Thus all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of the tree T. By Observation 1 we have $y \in D$. Let us observe that $D \setminus \{y\}$ is an m2ds of the tree T' as the vertex x is still dominated at least twice. By the inductive hypothesis we have $D \setminus \{y\} \in \mathcal{F}(T')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get $|\mathcal{F}(T)| = |\mathcal{F}(T')| \leq \alpha^{n-1} < \alpha^n$. Henceforth, we can assume that every support vertex of T is adjacent to at most two leaves.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that $d_T(v) = 3$. By a we denote the leaf adjacent to v and different from t. Let $T' = T - T_v$ and $T'' = T - t - a$. Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of the tree T. By Observation 1 we have $t, a \in D$. If $v \notin D$, then observe that $D \setminus \{t, a\}$ is an m2ds of the tree T'. By the inductive hypothesis we have $D \setminus \{t, a\} \in \mathcal{F}(T')$. Now assume that $v \in D$. Let us observe that $D \setminus \{t, a\}$ is an m2ds of the tree T''. By the inductive hypothesis we have $D \setminus \{t, a\} \in \mathcal{F}(T'')$. The set $D \setminus \{v, t, a\}$ is not an m2ds of the tree T', otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree T, a contradiction to the minimality of D. By the inductive hypothesis we have $D \setminus \{v, t, a\} \notin \mathcal{F}(T')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get $|\mathcal{F}(T)| = |\mathcal{F}(T')| + |D'' \in \mathcal{F}(T'') \colon D'' \setminus \{v\} \notin \mathcal{F}(T')| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-3} + \alpha^{n-2} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

Now assume that $d_T(v) = 2$. Assume that $d_T(u) \geq 3$. Let $T' = T - T_v$ and $T'' = T - T_u$. Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of the tree T. By Observation 1 we have $t \in D$. If $v \notin D$, then $u \in D$ as the vertex v has to be dominated twice. Observe that $D \setminus \{t\}$ is an m2ds of the tree T'. By the inductive hypothesis we have $D \setminus \{t\} \in \mathcal{F}(T')$. Now assume that $v \in D$. We have $u \notin D$, otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree T, a contradiction to the minimality of D. Observe that $D \cap V(T'')$ is an m2ds of the tree T''. By the inductive hypothesis we have $D \cap V(T'') \in \mathcal{F}(T'')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get $|\mathcal{F}(T)| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_v$ and $T'' = T - T_u$. Let us observe that all elements of $\mathcal{F}(T)$ are minimal 2-dominating sets of the tree T. Now let D be any m2ds of the tree T. By Observation 1 we have $t \in D$. If $v \notin D$, then observe that $D \setminus \{t\}$ is an m2ds of the tree T'. By the inductive hypothesis we have $D \setminus \{t\} \in \mathcal{F}(T')$. Now assume that $v \in D$. We have $u \notin D$, otherwise $D \setminus \{v\}$ is a 2-dominating set of the tree T, a contradiction to the minimality of D. Moreover, we have $w \in D$ as the vertex u has to be dominated twice. Observe that $D \setminus \{v, t\}$ is an m2ds of the tree T''. By the inductive hypothesis we have $D \setminus \{v, t\} \in \mathcal{F}(T'')$. Therefore $\mathcal{F}(T)$ contains all minimal 2-dominating sets of the tree T. Now we get $|\mathcal{F}(T)| \leq |\mathcal{F}(T')| + |\mathcal{F}(T'')| \leq \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-3}(\alpha + 1) = \alpha^{n-3} \cdot \alpha^3 = \alpha^n$. ■

It follows from the proof of the above theorem that any tree of order n has at most 1.3248^n minimal 2-dominating sets.

Corollary 3 *Every tree of order n has at most α^n minimal 2-dominating sets, where $\alpha \approx 1.3248$ is the positive solution of the equation $x^3 - x - 1 = 0$.*

Now we show that the bound from the previous corollary is tight. Let a_n denote the number of minimal 2-dominating sets of the path P_n. The next remark follows from the proof of Theorem 2.

Remark 4 *For every positive integer n we have*

$$a_n = \begin{cases} 1 & \text{if } n \leq 3; \\ a_{n-3} + a_{n-2} & \text{if } n \geq 4. \end{cases}$$

We have $\lim_{n\to\infty} \sqrt[n]{a_n} = \alpha$, where $\alpha \approx 1.3247$ is the positive solution of the equation $x^3 - x - 1 = 0$. This implies that the bound from Corollary 3 is tight.

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