Non-isolating bondage in graphs

Marcin Krzywkowski*†
marcin.krzywkowski@gmail.com

Abstract

A dominating set of a graph $G = (V, E)$ is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The non-isolating bondage number of G, denoted by $b'(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma(G - E') > \gamma(G)$. If for every $E' \subseteq E$ we have $\gamma(G - E') = \gamma(G)$ or $\delta(G - E') = 0$, then we define $b'(G) = 0$, and we say that G is a γ-non-isolatingly strongly stable graph. First we discuss various properties of non-isolating bondage in graphs. We find the non-isolating bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating bondage number. Finally, we characterize all γ-non-isolatingly strongly stable trees.

Keywords: domination, bondage, non-isolating bondage, graph, tree.
$\mathcal{AMS}$ **Subject Classification:** 05C05, 05C69.

1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. Let $\delta(G)$ mean the minimum degree among all vertices of G. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The distance between two vertices of a graph is the number of edges

*Research fellow at the Department of Mathematics, University of Johannesburg, South Africa.
†Faculty of Electronics, Telecommunications and Informatics, Gdansk University of Technology, Poland. Research partially supported by the Polish National Science Centre grant 2011/02/A/ST6/00201.

in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. We denote the path (cycle, respectively) on n vertices by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, of degree two such that the tree resulting from T by removing the edge vx, and which contains the vertex x, is a path P_n. Let $K_{p,q}$ denote a complete bipartite graph the partite sets of which have cardinalities p and q. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set, abbreviated DS, of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see for example [5].

The bondage number $b(G)$ of a graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma(G - E') > \gamma(G)$. The concept of bondage in graphs was introduced in [2], and further studied for example in [1, 3, 4, 6–9].

We define the non-isolating bondage number of a graph G, denoted by $b'(G)$, to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G-E') \geq 1$ and $\gamma(G - E') > \gamma(G)$. Thus $b'(G)$ is the minimum number of edges of G that have to be removed in order to obtain a graph with no isolated vertices, and with the domination number greater than that of G. If for every $E' \subseteq E$ we have $\gamma(G - E') = \gamma(G)$ or $\delta(G - E') = 0$, then we define $b'(G) = 0$, and we say that G is a γ-non-isolatingly strongly stable graph.

First we discuss various properties of non-isolating bondage in graphs. We find the non-isolating bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree having such non-isolating bondage number. Finally, we characterize all γ-non-isolatingly strongly stable trees.

2 Results

We begin with the following well known observations.

For every graph G of diameter at least two there exists a $\gamma(G)$-set that contains all support vertices.

If H is a subgraph of G such that $V(H) = V(G)$, then $\gamma(H) \geq \gamma(G)$.

If n is a positive integer, then $\gamma(P_n) = \lfloor (n + 2)/3 \rfloor$.

For every integer $n \geq 3$ we have $\gamma(C_n) = \lfloor (n + 2)/3 \rfloor$.

Observation 1 *If n is a positive integer, then $\gamma(K_n) = 1$.*

Observation 2 *For every integer $n \geq 4$ we have $\gamma(W_n) = 1$.*

Observation 3 *Let p and q be positive integers such that $p \leq q$. Then*

$$\gamma(K_{p,q}) = \begin{cases} 1 & \text{if } p = 1; \\ 2 & \text{otherwise.} \end{cases}$$

First we calculate the non-isolating bondage numbers of paths.

Lemma 4 *For any positive integer n we have*

$$b'(P_n) = \begin{cases} 0 & \text{if } n = 1,2,3,4,5,7; \\ 1 & \text{if } n \geq 6 \text{ and } n \neq 3k+1; \\ 2 & \text{if } n \geq 10 \text{ and } n = 3k+1. \end{cases}$$

Proof. Let us observe that if a path has at most five or exactly seven vertices, then removing any edges does not increase the domination number, or gives an isolated vertex. Assume that $n = 6$ or $n \geq 8$. First assume that $n = 3k$. We have $\gamma(P_n) = \lfloor (n+2)/3 \rfloor = \lfloor (3k+2)/3 \rfloor = k$. We also have $\gamma(P_{n-2}) + \gamma(P_2)$ $= \lfloor n/3 \rfloor + 1 = k + 1 > \gamma(P_n)$. Thus $b'(P_n) = 1$ if $n = 3k$ and $n \geq 6$. Now assume that $n = 3k+2$. We have $\gamma(P_n) = \lfloor (n+2)/3 \rfloor = \lfloor (3k+4)/3 \rfloor = k+1$. We also have $\gamma(P_{n-4}) + \gamma(P_4) = \lfloor n/3 \rfloor + 2 = k+2 > \gamma(P_n)$. Thus $b'(P_n) = 1$ if $n = 3k+2$ and $n \geq 8$. Now assume that $n = 3k+1$. We have $\gamma(P_n) = \lfloor (n+2)/3 \rfloor$ $= \lfloor (3k+3)/3 \rfloor = k+1$. Let us observe that removing any edge does not increase the domination number. We have $\gamma(P_{n-6}) + \gamma(P_4) + \gamma(P_2) = \lfloor (n-4)/3 \rfloor + 3$ $= \lfloor (3k-3)/3 \rfloor + 3 = k+2 > \gamma(P_n)$. Therefore $b'(P_n) = 2$ if $n = 3k+1$ and $n \geq 10$. ∎

We now investigate the non-isolating bondage in cycles.

Lemma 5 *For every integer $n \geq 3$ we have*

$$b'(C_n) = \begin{cases} 0 & \text{if } b'(P_n) = 0; \\ b'(P_n) + 1 & \text{if } b'(P_n) \neq 0. \end{cases}$$

Proof. We have $\gamma(P_n) = \gamma(C_n)$. Clearly, $C_n - e = P_n$. This implies that $b'(C_n) = 0$ if $b'(P_n) = 0$, while $b'(C_n) = b'(P_n) + 1$ if $b'(P_n) \neq 0$. ∎

We now find the non-isolating bondage numbers of complete graphs.

Proposition 6 *If n is a positive integer, then*

$$b'(K_n) = \begin{cases} 0 & \text{for } n = 1,2,3; \\ \lfloor (n+1)/2 \rfloor & \text{for } n \geq 4. \end{cases}$$

Proof. Obviously, $b'(K_1) = 0$ and $b'(K_2) = 0$. We have $K_3 - e = C_3$ and $b'(C_3)$ $= 0$. This implies that $b'(K_3) = 0$. Now assume that $n \geq 4$. By Observation 1 we have $\gamma(K_n) = 1$. Let us observe that the domination number of a graph equals one if and only if the graph has a universal vertex. Given a complete graph, we increase the domination number if and only if for every vertex we remove at least one incident edge. If n is even, then we remove $n/2 = \lfloor (n+1)/2 \rfloor$ edges. If n is odd, then we remove $(n-1)/2 + 1 = (n+1)/2 = \lfloor (n+1)/2 \rfloor$ edges. ■

We now calculate the non-isolating bondage numbers of wheels.

Proposition 7 *For integers $n \geq 4$ we have*

$$b'(W_n) = \begin{cases} 2 & \text{if } n = 4; \\ 1 & \text{if } n \geq 5. \end{cases}$$

Proof. Since $W_4 = K_4$, using Proposition 6 we get $b'(W_4) = b'(K_4) = \lfloor 5/2 \rfloor = 2$. Now assume that $n \geq 5$. By Observation 2 we have $\gamma(W_n) = 1$. The domination number of a graph equals one if and only if it has a universal vertex. Removing an edge of W_n incident to the vertex of maximum degree gives a graph without universal vertices. Therefore $b'(W_n) = 1$ for $n \geq 5$. ■

We now investigate the non-isolating bondage in complete bipartite graphs.

Proposition 8 *Let p and q be positive integers such that $p \leq q$. Then*

$$b'(K_{p,q}) = \begin{cases} 0 & \text{if } p = 1, 2; \\ 4 & \text{if } p = 3; \\ p & \text{otherwise.} \end{cases}$$

Proof. Let $E(K_{p,q}) = \{a_i b_j \colon 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$. If $p = 1$, then obviously $b'(K_{p,q}) = 0$ as removing any edge produces an isolated vertex. Now assume that $p \geq 2$. By Observation 3 we have $\gamma(K_{p,q}) = 2$. Let E' be a subset of the set of edges of $K_{2,q}$ such that $\delta(K_{2,q} - E') \geq 1$. Each vertex b_i is adjacent to a_1 or a_2 in the graph $K_{2,q} - E'$. Observe that the vertices a_1 and a_2 form a dominating set of $K_{2,q} - E'$. Therefore $b'(K_{2,q}) = 0$. Now assume that $p = 3$. It is not very difficult to verify that removing any three edges does not increase the domination number while not producing an isolated vertex. We have $\gamma(K_{3,q} - a_1b_2 - a_1b_3$ $-a_2b_1 - a_3b_1) = 3 > 2 = \gamma(K_{3,q})$. Therefore $b'(K_{3,q}) = 4$. Now assume that $p \geq 4$. If we remove at most $p - 1$ edges, then there are vertices a_i and b_j which have degrees q and p, respectively. It is easy to observe that the vertices a_i and b_j still form a dominating set. Let us observe that $\gamma(K_{p,q} - a_1b_1 - a_2b_1 - a_3b_2$ $-a_4b_2 - a_5b_2 - \ldots - a_pb_2) = 3 > 2 = \gamma(K_{p,q})$. Therefore $b'(K_{p,q}) = p$ if $p \geq 4$. ■

The authors of [2] proved that the bondage number of any tree is either one or two.

Theorem 9 ([2]) *For every tree T we have $b(T) \in \{1, 2\}$.*

Let us observe that for every non-negative integer there exists a tree with such non-isolating bondage number. We have $b'(P_4) = 0$. For positive integers k, consider trees T_k of the form presented in Figure 1. It is not difficult to verify that $b'(T_k) = k$.

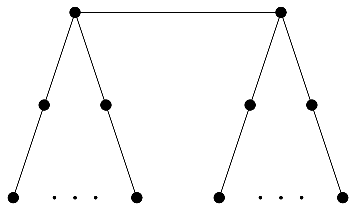

Figure 1: A tree T_k having $4k + 2$ vertices, where both central vertices are of degree $k + 1$

Hartnell and Rall [3] characterized all trees with bondage number equal to two. We characterize all trees with the non-isolating bondage number equal to zero, that is, all γ-non-isolatingly strongly stable trees.

We now show that joining two γ-non-isolatingly strongly stable trees gives us also a γ-non-isolatingly strongly stable tree.

Lemma 10 *Let T_1 and T_2 be vertex-disjoint γ-non-isolatingly strongly stable trees. Let x be a support vertex of T_1 and let y be a leaf of T_2. Let T be a tree obtained by joining the vertices x and y. If $\gamma(T) = \gamma(T_1) + \gamma(T_2)$, then the tree T is also γ-non-isolatingly strongly stable.*

Proof. Let E_1 be a subset of the set of edges of T such that $\delta(T - E_1) \geq 1$. If $xy \in E_1$, then we get $\gamma(T - E_1) = \gamma(T_1 - E_1 \cap E(T_1)) + \gamma(T_2 - E_1 \cap E(T_2))$ $= \gamma(T_1) + \gamma(T_2) = \gamma(T)$. Now assume that $xy \notin E_1$. Let z be the neighbor of y other than x. If $yz \notin E_1$, then let $E_2 = E_1 \cup \{xy\}$. Similarly as earlier we get $\gamma(T - E_2) = \gamma(T)$. We have $\gamma(T - E_1) \leq \gamma(T - E_2)$, and consequently, $\gamma(T - E_1) = \gamma(T)$. Now assume that $yz \in E_1$. Let $E_3 = E_1 \cup \{xy\} \setminus \{yz\}$. Similarly as earlier we get $\gamma(T - E_3) = \gamma(T)$. Let D_2 be a $\gamma(T - E_3)$-set that contains the vertices x and z. It is easy to observe that D_2 is also a DS of the graph $T - E_1$. Therefore $\gamma(T - E_1) \leq \gamma(T - E_3)$. This implies that $\gamma(T - E_1) = \gamma(T)$. We now conclude that $b'(T) = 0$. ∎

We next show that a subtree of a γ-non-isolatingly strongly stable tree is also γ-non-isolatingly strongly stable.

Lemma 11 *Let T be a γ-non-isolatingly strongly stable tree. Assume that T' is a subtree of T such that $T - T'$ has no isolated vertices. Then $b'(T') = 0$.*

Proof. If T' consists of a single vertex, then obviously $b'(T') = 0$. Thus assume that $T' \neq K_1$. Let E_1 be the minimum subset of $E(T)$ such that T' is a component of $T - E_1$. Now let E' be a subset of $E(T')$ such that $\delta(T' - E') \geq 1$. Notice that $\delta(T - E_1 - E') \geq 1$. The assumption $b'(T) = 0$ implies that $\gamma(T - E_1) = \gamma(T)$ and $\gamma(T - E_1 - E') = \gamma(T)$. We have $T - E_1 - E' = T' - E' \cup (T - T')$ and $T - E_1 = T' \cup (T - T')$. We now get $\gamma(T' - E') = \gamma(T - E_1 - E') - \gamma(T - T')$ $= \gamma(T) - \gamma(T - E_1) + \gamma(T') = \gamma(T')$. This implies that $b'(T') = 0$. ■

For the purpose of characterizing all γ-non-isolatingly strongly stable trees, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_1, P_2\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_k, which is adjacent to a path P_1 or P_4, or is not a leaf and is adjacent to a support vertex.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_1 or P_3.
- Operation $\mathcal{O}_4$: Attach a path P_5 by joining one of its leaves to any support vertex of T_k.

We now prove that every tree of the family $\mathcal{T}$ is γ-non-isolatingly strongly stable.

Lemma 12 *If $T \in \mathcal{T}$, then $b'(T) = 0$.*

Proof. We use induction on the number k of operations performed to construct the tree T. If $T = P_1$, then obviously $b'(T) = 0$. If $T = P_2$, then $b'(T) = 0$ as removing the edge gives isolated vertices. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by Operation $\mathcal{O}_1$. Let x be the attached vertex, and let y be its neighbor. Let z be a leaf adjacent to y and different from x. Let D be a $\gamma(T)$-set that contains all support vertices. The set D is minimal, thus $x \notin D$. Obviously, D is a DS of the tree T'. Therefore $\gamma(T')$ $\leq \gamma(T)$. Now let E' be a subset of the set of edges of T such that $\delta(T - E') \geq 1$. Since both x and z are leaves of T, we have $xy \notin E'$ and $yz \notin E'$. The assumption $b'(T') = 0$ implies that $\gamma(T' - E') = \gamma(T')$. Let us observe that there exists a $\gamma(T' - E')$-set that contains the vertex y. Let D' be such a set. It is easy to see that D' is a DS of the graph $T - E'$. Thus $\gamma(T - E') \leq \gamma(T' - E')$.

We now get $\gamma(T-E') \leq \gamma(T'-E') = \gamma(T') \leq \gamma(T)$. On the other hand, we have $\gamma(T-E') \geq \gamma(T)$. This implies that $\gamma(T-E') = \gamma(T)$, and consequently, $b'(T) = 0$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_2$. The vertex to which is attached P_2 we denote by x. Let v_1v_2 be the attached path. Let v_1 be joined to x. If x is adjacent to a leaf or a support vertex, say a, then let D be a $\gamma(T)$-set that contains all support vertices. We have $v_2 \notin D$ as the set D is minimal. It is easy to observe that $D \setminus \{v_1\}$ is a DS of the tree T'. If x is adjacent to a path P_4, then we denote it by $abcd$. Let a and x be adjacent. Let us observe that there exists a $\gamma(T)$-set that contains the vertices v_1, c, and x. Let D be such a set. It is easy to observe that $D \setminus \{v_1\}$ is a DS of the tree T'. We conclude that $\gamma(T') \leq \gamma(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T-E') \geq 1$. Since v_2 is a leaf of T, we have $v_1v_2 \notin E'$. If $xv_1 \in E'$, then $\delta(T' - (E' \cap E(T'))) \geq 1$. We get $\gamma(T-E') = \gamma(P_2 \cup T' - (E' \setminus \{xv_1\}))$ $= \gamma(T' - (E' \cap E(T'))) + \gamma(P_2) = \gamma(T') + 1 \leq \gamma(T)$. Now assume that $xv_1 \notin E'$. By T_x (T'_x, respectively) we denote the component of $T-E'$ ($T'-E'$, respectively) which contains the vertex x. If $\delta(T'-(E' \cap E(T'))) \geq 1$, then let D'_x be any $\gamma(T'_x)$-set. It is easy to see that $D'_x \cup \{v_1\}$ is a DS of the tree T_x. Thus $\gamma(T_x) \leq \gamma(T'_x)+1$. We now get $\gamma(T-E') = \gamma(T-E'-T_x) + \gamma(T_x) \leq \gamma(T-E'-T_x) + \gamma(T'_x) + 1$ $= \gamma(T'-E'-T'_x) + \gamma(T'_x) + 1 = \gamma(T'-E') + 1 = \gamma(T') + 1 \leq \gamma(T)$. Now assume that $\delta(T' - (E' \cap E(T'))) = 0$. This implies that x is the only isolated vertex of $T' - (E' \cap E(T'))$, and so x is not adjacent to any leaf in the trees T' and T. Consequently, T'_x consists only of the vertex x, and T_x is a path P_3. Let us observe that $\delta(T' - (E' \setminus \{xa\})) \geq 1$. Let T'_a be the component of $T'-E'$, which contains the vertex a. Now let T''_a be a tree obtained from T'_a by attaching a vertex to the vertex a. We now get $\gamma(T-E') = \gamma(T-E'-T_x) + \gamma(P_3) = \gamma(T'-E'-T'_x) + 1$ $= \gamma(T'-E'-T'_x-T'_a) + \gamma(T'_a) + 1 \leq \gamma(T'-E'-T'_x-T'_a) + \gamma(T''_a) + 1 = \gamma((T'-E'$ $-T'_x-T'_a) \cup T''_a) + 1 = \gamma(T'-(E' \setminus \{xa\})) + 1 = \gamma(T'-E') + 1 = \gamma(T') + 1 \leq \gamma(T)$. We conclude that $\gamma(T-E') = \gamma(T)$, and consequently, $b'(T) = 0$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_3$. The vertex to which is attached P_3 we denote by x. If x is a support vertex, then using Lemma 10, for $T_1 = T'$ and $T_2 = P_3$, we get $b'(T) = 0$. Now assume that x is adjacent to a path P_3, say abc. Let a and x be adjacent. The attached path we denote by $v_1v_2v_3$. Let v_1 be joined to x. Let us observe that there exists a $\gamma(T)$-set that contains all support vertices, and does not contain the vertex v_1. Let D be such a set. We have $v_3 \notin D$ as the set D is minimal. Observe that $D \setminus \{v_2\}$ is a DS of the tree T'. Therefore $\gamma(T') \leq \gamma(T) - 1$. Now let E' be a subset of the set of edges of T such that $\delta(T-E') \geq 1$. We have $v_2v_3 \notin E'$ as the vertex v_3 is a leaf. If $xv_1 \in E'$, then $v_1v_2 \notin E'$; otherwise we get an isolated vertex. Let us observe that $\delta(T'-(E' \cap E(T'))) \geq 1$. We get $\gamma(T-E')$ $= \gamma(P_3 \cup T - (E' \setminus \{xv_1\})) = \gamma(T' - (E' \cap E(T'))) + \gamma(P_3) = \gamma(T') + 1 \leq \gamma(T)$. Now assume that $xv_1 \notin E'$. Because of the similarity between the paths abc and $v_1v_2v_3$ adjacent to the vertex x, it suffices to consider only the possibility when $xa \notin E'$. Let us observe that $\delta(T' - (E' \cap E(T'))) \geq 1$. By T_x (T'_x, respectively) we denote the component of $T - E'$ ($T' - (E' \cap E(T'))$, respectively) which contains the vertex x. If $v_1v_2 \notin E'$, then let D'_x be any $\gamma(T'_x)$-set. It is easy to see that $D'_x \cup \{v_2\}$ is a DS of the tree T_x. Thus $\gamma(T_x) \leq \gamma(T'_x) + 1$. We now get $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x) \leq \gamma(T - E' - T_x) + \gamma(T'_x) + 1$ $= \gamma(T' - E' - T'_x) + \gamma(T'_x) + 1 = \gamma(T' - E') + 1 = \gamma(T') + 1 \leq \gamma(T)$. Now assume that $v_1v_2 \in E'$. Because of the similarity between the paths abc and $v_1v_2v_3$, it suffices to consider only the possibility when $ab \in E'$. Let D'_x be a $\gamma(T'_x)$-set that contains all support vertices (so $x \in D'_x$). It is easy to see that D'_x is a DS of the tree T_x. Thus $\gamma(T_x) \leq \gamma(T'_x)$. We get $\gamma(T - E') = \gamma(T - E' - T_x) + \gamma(T_x)$ $\leq \gamma(T - E' - T_x) + \gamma(T'_x) = \gamma(T' - E' - T'_x) + \gamma(T'_x) = \gamma(T' - E') = \gamma(T') \leq \gamma(T)$. We now conclude that $\gamma(T - E') = \gamma(T)$, and consequently, $b'(T) = 0$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_4$. By Lemma 4 we have $b'(P_5) = 0$. Using Lemma 10, for $T_1 = T'$ and $T_2 = P_5$, we get $b'(T) = 0$. ■

We now prove that if a tree is γ-non-isolatingly strongly stable, then it belongs to the family $\mathcal{T}$.

Lemma 13 *Let T be a tree. If $b'(T) = 0$, then $T \in \mathcal{T}$.*

Proof. If $\text{diam}(T) \in \{0, 1\}$, then $T \in \{P_1, P_2\} \subseteq \mathcal{T}$. If $\text{diam}(T) = 2$, then T is a star. The tree T can be obtained from P_2 by an appropriate number of Operations $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be a $\gamma(T')$-set that contains all support vertices. It is easy to see that D' is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')$. Now let E' be a subset of the set of edges of T' such that $\delta(T' - E') \geq 1$. Since $b'(T) = 0$, we have $\gamma(T - E') = \gamma(T)$. Let us observe that there exists a $\gamma(T - E')$-set that contains the vertex x. Let D be such a set. The set D is minimal, thus $y \notin D$. Obviously, D is a DS of the graph $T' - E'$. Therefore $\gamma(T' - E') \leq \gamma(T - E')$. We now get $\gamma(T' - E') \leq \gamma(T - E') = \gamma(T) \leq \gamma(T')$. On the other hand, we have $\gamma(T' - E') \geq \gamma(T')$. This implies that $\gamma(T' - E') = \gamma(T')$, and consequently, $b'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u. If $\text{diam}(T) \geq 5$, then let d be the parent of w. If $\text{diam}(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that $d_T(u) \geq 3$. Thus some child of u is a leaf or a support vertex other than v. Let $T' = T - T_v$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geq 3$. First assume that there is a child of w other than u, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_u$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that some child of w is a leaf. Let $T' = T - T_u$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Thus there is a child of w, say k, such that the distance of w to the most distant vertex of T_k is two. Consequently, k is a support vertex of degree two. Due to the earlier analysis of the children of the vertex u, it suffices to consider only the possibility when $d_T(w) = 3$. Let $T' = T - T_w$. It is easy to observe that $D' \cup \{v, k\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 2$. We have $\delta(T - dw - uv - wk) \geq 1$. We now get $\gamma(T - dw - uv - wk) = \gamma(T' \cup P_2 \cup P_2 \cup P_2) = \gamma(T') + 3\gamma(P_2)$ $= \gamma(T') + 3 \geq \gamma(T) + 1 > \gamma(T)$. This implies that $b'(T) \neq 0$, a contradiction.

If $d_T(w) = 1$, then $T = P_4$. Let $T' = P_2 \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. Now assume that $d_T(w) = 2$. First assume that there is a child of d other than w, say k, such that the distance of d to the most distant vertex of T_k is four or one. It suffices to consider only the possibilities when T_k is a path P_4, or k is a leaf. Let $T' = T - T_w$. Let us observe that there exists a $\gamma(T')$-set that contains the vertex d. Let D' be such a set. It is easy to observe that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. We have $\delta(T - dw - uv) \geq 1$. We now get $\gamma(T - dw - uv) = \gamma(T' \cup P_2 \cup P_2)$ $= \gamma(T') + 2\gamma(P_2) = \gamma(T') + 2 \geq \gamma(T) + 1 > \gamma(T)$. This implies that $b'(T) \neq 0$, a contradiction.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_l$. Due to the similarity of T' to the tree T from the previous case when d is adjacent to a leaf, we conclude that $b'(T') \neq 0$. On the other hand, by Lemma 11 we have $b'(T') = 0$, a contradiction.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. Let $T' = T - T_k$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

If $d_T(d) = 1$, then $T = P_5$. Let $T' = P_2 \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(d) = 2$. First assume that e is adjacent to a leaf, say k.

Let $T' = T - T_d$. By Lemma 11 we have $b'(T') = 0$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that e is not adjacent to any leaf. Let E' be the set of edges incident with e excluding ed. Let $G' = T - T_d - e$. Let D' be any $\gamma(G')$-set. It is easy to observe that $D' \cup \{d, v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(G') + 2$. We have $\delta(T - (E' \cup \{dw, uv\})) \geq 1$. We now get $\gamma(T - (E' \cup \{dw, uv\})) = \gamma(G' \cup P_2 \cup P_2 \cup P_2) = \gamma(G') + 3\gamma(P_2) = \gamma(G') + 3 \geq \gamma(T) + 1 > \gamma(T)$. This implies that $b'(T) \neq 0$, a contradiction. ■

As an immediate consequence of Lemmas 12 and 13, we have the following characterization of all γ-non-isolatingly strongly stable trees.

Theorem 14 *Let T be a tree. Then $b'(T) = 0$ if and only if $T \in \mathcal{T}$.*

References

[1] G. Domke and R. Laskar, *The bondage and reinforcement numbers of γ_f for some graphs,* Discrete Mathematics 167/168 (1997), 249–259.

[2] J. Fink, M. Jacobson, L. Kinch and J. Roberts, *The bondage number of a graph,* Discrete Mathematics 86 (1990), 47–57.

[3] B. Hartnell and D. Rall, *A characterization of trees in which no edge is essential to the domination number,* Ars Combinatoria 33 (1992), 65–76.

[4] B. Hartnell and D. Rall, *Bounds on the bondage number of a graph,* Discrete Mathematics 128 (1994), 173–177.

[5] T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

[6] L. Kang and J. Yuan, *Bondage number of planar graphs,* Discrete Mathematics 222 (2000), 191–198.

[7] M. Krzywkowski, *2-bondage in graphs,* International Journal of Computer Mathematics 90 (2013), 1358–1365.

[8] H. Liu and L. Sun, *The bondage and connectivity of a graph,* Discrete Mathematics 263 (2003), 289–293.

[9] U. Teschner, *New results about the bondage number of a graph,* Discrete Mathematics 171 (1997), 249–259.