Bulletin of the Iranian Mathematical Society Vol. XX *No. X (201X), pp XX-XX.*

ON TREES ATTAINING AN UPPER BOUND ON THE TOTAL DOMINATION NUMBER

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Communicated by

ABSTRACT. A total dominating set of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Chellali and Haynes [*Total and paired-domination numbers of a tree,* AKCE International Journal of Graphs and Combinatorics 1 (2004), 69–75] established the following upper bound on the total domination number of a tree in terms of the order and the number of support vertices, $\gamma_t(T) \leq (n+s)/2$. We characterize all trees attaining this upper bound.

1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong

MSC(2010): Primary: 05C69; Secondary: 05C05.
Keywords: domination, total domination, tree.
Received: ?? Month 20??, Accepted: ?? Month 20??.

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(weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G)\setminus D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see [6, 7].

A subset $D \subseteq V(G)$ is a total dominating set, abbreviated TDS, of G if every vertex of G has a neighbor in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of G of minimum cardinality is called a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3], and further studied for example in [1, 5, 8].

Chellali and Haynes [2] established the following upper bound on the total domination number of a tree, $\gamma_t(T) \leq (n+s)/2$, where n means the order and s means the number of support vertices of the tree T.

DeLa Viña et al. [4] improved the above bound. They proved that if T is a tree different from star, then $\gamma_t(T) \leq (n+s)/2 - (l-s^*)/2$, where l means the number of leaves and s^* means the number of support vertices non-adjacent to any other support vertex.

We characterize all trees attaining the upper bound of Chellali and Haynes.

2. Results

Since the one-vertex graph does not have a total dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

Observation 2.1. *Every support vertex of a graph G is in every TDS of G.*

Observation 2.2. *For every connected graph G of diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaf.*

Chellali and Haynes [2] proved that for every tree T of order $n \geq 3$ with s support vertices we have $\gamma_t(T) \leq (n+s)/2$. It is easy to see that the path P_2 also satisfies the inequality. Therefore we have the following result.

Theorem 2.3. ([2]) *For every tree T of order n with s support vertices we have $\gamma_t(T) \leq (n+s)/2$.*

To characterize the trees attaining the bound from the previous theorem, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3\}$. If $T_1 = P_2$, then the vertices of T_1 we denote by x and y. If $T_1 = P_3$, then the support vertex of T_1 we denote by y, and one of the leaves we denote by x. Let $A(T_1) = \{x, y\}$. Now let H_1 be a path P_3 with the support vertex labeled v and one of the leaves labeled u. Let H_2 be a path P_4 with the support vertices labeled u and v. The leaf adjacent to u we denote by t, and the leaf adjacent to v we denote by w. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a copy of H_1 by joining the vertex u to a vertex of T_k adjacent to a path P_3. Let $A(T) = A(T') \cup \{u, v\}$.
- Operation $\mathcal{O}_2$: Attach a copy of H_1 by joining the vertex u to a vertex of T_k which is not a leaf and is adjacent to a support vertex. Let $A(T) = A(T') \cup \{u, v\}$.
- Operation $\mathcal{O}_3$: Attach a copy of H_2 by joining the vertex t to a leaf of T_k adjacent to a weak support vertex. Let $A(T) = A(T')$ $\cup\{u, v\}$.

Note that for the path P_2, only operation $\mathcal{O}_3$ can be applied. Both vertices of P_2 are leaves and at the same time they are weak support vertices.

We now prove that for every tree T of the family $\mathcal{T}$, the set $A(T)$ defined above is a TDS of minimum cardinality equal to $(n+l)/2$.

Lemma 2.3. *If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_t(T)$-set of size $(n+s)/2$.*

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graphs H_1 and H_2 defined above. To show that $A(T)$ is a $\gamma_t(T)$-set of cardinality $(n+s)/2$, we use the induction on the number k of operations performed to construct the tree T. If $T = P_2$, then $(n+s)/2 = (2+2)/2 = 2 = |A(T)| = \gamma_t(T)$. If $T = P_3$, then $(n+s)/2 = (3+1)/2 = 2 = |A(T)| = \gamma_t(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. For a given tree T', let n' denote its order and s' the number of its support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n' + 3$ and $s = s' + 1$. The vertex to which is attached P_3 we denote by x. Let abc denote a path P_3 adjacent to x, and such that $a \neq u$. Let a and x be adjacent. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 2.1 we have $v \in D$. Each one of the vertices v and b has to have a neighbor in D, thus $u, a \in D$. Let us observe that $D \setminus \{u, v\}$ is a TDS of the tree T' as the vertex x has a neighbor in $D \setminus \{u, v\}$. Therefore, $\gamma_t(T') \leq \gamma_t(T) - 2$. We now conclude that $\gamma_t(T) = \gamma_t(T') + 2$. We get $\gamma_t(T) = |A(T)| = |A(T')| + 2 = (n' + s')/2 + 2 = (n - 3 + s - 1)/2 + 2 = (n + s)/2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n' + 3$ and $s = s' + 1$. The vertex to which is attached P_3 we denote by x. Let y be a support vertex adjacent to x. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_t(T)$-set that contains no leaf. By Observation 2.1 we have $v, y \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. It is easy to observe that $D \setminus \{u, v\}$ is a TDS of the tree T'. Therefore, $\gamma_t(T') \leq \gamma_t(T) - 2$. We now conclude that $\gamma_t(T) = \gamma_t(T') + 2$. We get $\gamma_t(T) = |A(T)| = |A(T')| + 2 = (n' + s')/2 + 2 = (n - 3 + s - 1)/2 + 2 = (n + s)/2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n' + 4$ and $s = s'$. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let us observe that there exists a $\gamma_t(T)$-set that does not contain the vertices w and t. Let D be such a set. By Observation 2.1 we have $v \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. Observe that $D \setminus \{u, v\}$ is a TDS of the tree T'. Therefore, $\gamma_t(T') \leq \gamma_t(T) - 2$. We now conclude that $\gamma_t(T) = \gamma_t(T') + 2$. We get $\gamma_t(T) = |A(T)| = |A(T')| + 2 = (n' + s')/2 + 2 = (n - 4 + s)/2 + 2 = (n + s)/2$. □

We now prove that if a tree attains the bound from Theorem 3, then the tree belongs to the family $\mathcal{T}$.

Lemma 2.4. *Let T be a tree of order n with s support vertices. If $\gamma_t(T) = (n + s)/2$, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the number of vertices of the tree T. If $\operatorname{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. Now assume that $\operatorname{diam}(T) = 2$. Thus

T is a star. If $T = P_3$, then $T \in \mathcal{T}$. If T is a star different from P_3, then $\gamma_t(T) = 2 < 5/2 \leq (n+1)/2 = (n+s)/2$.

Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. Assume that the lemma is true for every tree T' of order $n' < n$ with s' support vertices.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $n' = n - 1$ and $s' = s$. Let D' be any $\gamma_t(T')$-set. By Observation 2.1 we have $x \in D'$. Obviously, D' is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T')$. We now get $\gamma_t(T) \leq \gamma_t(T') \leq (n'+s')/2 = (n-1+s)/2 < (n+s)/2$, a contradiction. Thus every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. If $\mathrm{diam}(T) \geq 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) \geq 3$. Assume that among the children of u there is a support vertex, say x, different from v. Let $T' = T - T_v$. We have $n' = n - 2$ and $s' = s - 1$. Let D' be a $\gamma_t(T')$-set that contains no leaf. The vertex x has to have a neighbor in D', thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 1$. We now get $\gamma_t(T) \leq \gamma_t(T') + 1 \leq (n'+s')/2 + 1 = (n-2+s-1)/2 + 1 = (n+s)/2 - 1/2 < (n+s)/2$, a contradiction.

Thus, $d_T(u) = 3$ and the child of u other than v, say x, is a leaf. Let $T' = T - x$. We have $n' = n-1$ and $s' = s-1$. Let D' be a $\gamma_t(T')$-set that contains no leaf. The vertex v has to have a neighbor in D', thus $u \in D'$. It is easy to see that D' is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T')$. We now get $\gamma_t(T) \leq \gamma_t(T') \leq (n'+s')/2 = (n-1+s-1)/2 < (n+s)/2$, a contradiction.

Now assume that $d_T(u) = 2$. First assume that there is a child of w other than u, say x, such that the distance of w to the most distant vertex of T_x is three. It suffices to consider only the possibility when T_x is a path P_3. Let $T' = T - T_u$. We have $n' = n - 3$ and $s' = s - 1$. Let D' be any $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = (n+s)/2 - 2 = (n'+3+s'+1)/2 - 2 = (n'+s')/2$. This implies that $\gamma_t(T') = (n'+s')/2$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus, $T \in \mathcal{T}$.

Now assume that there is a child of w, say x, such that the distance of w to the most distant vertex of T_x is two. Thus x is a support vertex. Let $T' = T - T_u$. In the same way as in the previous possibility we conclude that $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus, $T \in \mathcal{T}$.

Now assume that some child of w, say x, is a leaf. It suffices to consider only the possibility when $d_T(w) = 3$. Let $T' = T - t - x$. We have $n' = n - 2$ and $s' = s - 1$. Let D' be a $\gamma_t(T')$-set that contains no leaf. By Observation 2.1 we have $u \in D'$. The vertex u has to have a neighbor in D', thus $w \in D'$. It is easy to observe that $D' \cup \{v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 1$. We now get $\gamma_t(T) \leq \gamma_t(T') + 1 \leq (n' + s')/2 + 1 = (n - 2 + s - 1)/2 + 1 = (n + s)/2 - 1/2 < (n + s)/2$, a contradiction.

If $d_T(w) = 1$, then $T = P_4$. We have $\gamma_t(T) = 2 < 3 = (4 + 2)/2 = (n+s)/2$, a contradiction. Now assume that $d_T(w) = 2$. First assume that $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $n' = n - 4$ and $s' = s - 1$. Let D' be any $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 2$. We now get $\gamma_t(T) \leq \gamma_t(T') + 2 \leq (n' + s')/2 + 2 = (n - 4 + s - 1)/2 + 2 = (n + s)/2 - 1/2 < (n + s)/2$, a contradiction.

If $d_T(d) = 1$, then $T = P_5$. We have $\gamma_t(T) = 3 < 7/2 = (5 + 2)/2 = (n + s)/2$, a contradiction. Now assume that $d_T(d) = 2$. Let $T' = T - T_w$. We have $n' = n-4$ and $s' \leq s$. Let D' be any $\gamma_t(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T. Thus, $\gamma_t(T) \leq \gamma_t(T') + 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = (n + s)/2 - 2 \geq (n' + 4 + s')/2 - 2 = (n' + s')/2$. This implies that $\gamma_t(T') = (n' + s')/2$ and $s' = s$. Therefore, $T' \in \mathcal{T}$ and the vertex e is not adjacent to any leaf in T. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus, $T \in \mathcal{T}$. □

As an immediate consequence of Lemmas 2.3 and 2.4, we have the following characterization of the trees attaining the bound from Theorem 2.3.

Theorem 2.5. *Let T be a tree. Then $\gamma_t(T) = (n + s)/2$ if and only if $T \in \mathcal{T}$.*

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