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2-bondage in graphs

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A 2-dominating set of a graph $G = (V, E)$ is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. The 2-bondage number of G, denoted by $b_2(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_2(G-E') > \gamma_2(G)$. If for every $E' \subseteq E$ we have $\gamma_2(G-E') = \gamma_2(G)$, then we define $b_2(G) = 0$, and we say that G is a γ_2-strongly stable graph. First we discuss the basic properties of 2-bondage in graphs. We find the 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree with such 2-bondage number. Finally, we characterize all trees with 2-bondage number equaling one or two.

Keywords: 2-domination; bondage; 2-bondage; tree

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1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong if it is adjacent to at least two leaves. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. The path (cycle, respectively) on n vertices is denoted by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of the cycle C_{n-1}. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Let $K_{p,q}$ denote a complete bipartite graph the partite sets of which have cardinalities p and q.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set, abbreviated as 2DS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. Note that 2-domination is a type of multiple domination in which each vertex, which is

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not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [3], and further studied for example in [1, 13]. For a comprehensive survey of domination in graphs, see [7, 8].

The bondage number $b(G)$ of a graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma(G - E') > \gamma(G)$. If for every $E' \subseteq E$ we have $\gamma(G-E') = \gamma(G)$, then we define $b(G) = 0$, and we say that G is a γ-strongly stable graph. Bondage in graphs was introduced in [4], and further studied for example in [2, 5, 6, 9–12, 14].

We define the 2-bondage number of G, denoted by $b_2(G)$, to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_2(G - E') > \gamma_2(G)$. Thus $b_2(G)$ is the minimum number of edges of G that have to be removed in order to increase the 2-domination number. If for every $E' \subseteq E$ we have $\gamma_2(G-E') = \gamma_2(G)$, then we define $b_2(G) = 0$, and we say that G is a γ_2-strongly stable graph.

First we discuss the basic properties of 2-bondage in graphs. We find the 2-bondage numbers for several classes of graphs. Next we show that for every non-negative integer there exists a tree with such 2-bondage number. Finally, we characterize all trees with 2-bondage number equaling one or two.

2. Results

We begin with the following observations.

Observation 2 Every leaf of a graph G is in every $\gamma_2(G)$-set.

Observation 3 If $H \subseteq G$ and $V(H) = V(G)$, then $\gamma_2(H) \geq \gamma_2(G)$.

Observation 4 For every positive integer n we have $\gamma_2(K_n) = \min\{2, n\}$.

Observation 5 If n is a positive integer, then $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.

Observation 6 For every integer $n \geq 3$ we have $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$.

Observation 7 For every integer $n \geq 4$ we have

$$\gamma_2(W_n) = \begin{cases} 2 & \text{if } n = 4, 5; \\ \lfloor (n+1)/3 \rfloor + 1 & \text{if } n \geq 6. \end{cases}$$

Observation 8 Let p and q be positive integers such that $p \leq q$. Then

$$\gamma_2(K_{p,q}) = \begin{cases} \max\{q, 2\} & \text{if } p = 1; \\ \min\{p, 4\} & \text{if } p \geq 2. \end{cases}$$

First we find the 2-bondage numbers of complete graphs.

Proposition 9 For every positive integer n we have

$$b_2(K_n) = \begin{cases} 0 & \text{if } n = 1, 2; \\ \lfloor 2n/3 \rfloor & \text{otherwise.} \end{cases}$$

Proof Obviously, $b_2(K_1) = 0$ and $b_2(K_2) = 0$. Now assume that $n \geq 3$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Observe that the 2-domination number of a graph equals two if and only if there is a pair of vertices, which are both adjacent to all vertices other than themselves. Let $E' \subseteq E(K_n)$. Let us observe $\gamma_2(K_n - E') > 2$ if and only if at most one vertex of K_n is not incident to any edge of E', and every edge of E' is adjacent to some other edge of E'. We want to choose a smallest set $E' \subseteq E(K_n)$ satisfying the above condition. Let us observe that the most efficient way is to choose for example the edges v_1v_2, v_2v_3, v_4v_5, v_5v_6, and so on. Let k be a positive integer. If $n = 3k$, then we have to remove $2k$ edges. Thus $b_2(K_{3k}) = 2k = 2n/3 = \lfloor 2n/3 \rfloor$. If $n = 3k+1$, then we also remove $2k$ edges as one vertex can remain universal. We have $b_2(K_{3k+1}) = 2k = \lfloor 2k + 2/3 \rfloor = \lfloor 2(3k+1)/3 \rfloor = \lfloor 2n/3 \rfloor$. Now assume that $n = 3k+2$. If we remove the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, \ldots, v_{3k-2}v_{3k-1}, v_{3k-1}v_{3k}$, then the vertices v_{3k+1} and v_{3k+2} remain universal. Therefore $b_2(K_{3k+2}) > 2k$. Let us observe that removing also the edge $v_{3k}v_{3k+1}$ suffices to increase the 2-domination number. This implies that $b_2(K_{3k+2}) = 2k+1 = \lfloor 2k + 4/3 \rfloor = \lfloor 2(3k+2)/3 \rfloor = \lfloor 2n/3 \rfloor$. ■

Now we calculate the 2-bondage numbers of paths.

Proposition 10 If n is a positive integer, then

$$b_2(P_n) = \begin{cases} 0 \text{ for } n = 1, 2; \\ 1 \text{ for } n \geq 3. \end{cases}$$

Now we investigate the 2-bondage in cycles.

Proposition 11 For every integer $n \geq 3$ we have

$$b_2(C_n) = \begin{cases} 1 \text{ if } n \text{ is even}; \\ 2 \text{ if } n \text{ is odd}. \end{cases}$$

Now we calculate the 2-bondage numbers of wheels.

Proposition 12 For every integer $n \geq 4$ we have

$$b_2(W_n) = \begin{cases} 1 \text{ if } n = 5; \\ 2 \text{ if } n \neq 3k+2; \\ 3 \text{ otherwise}. \end{cases}$$

Proof Let $E(W_n) = \{v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_2\}$. Using Proposition 9 we get $b_2(W_4) = b_2(K_4) = 2$. By Observation 7 we have $\gamma_2(W_5) = 2$. We also have $\gamma_2(W_5 - v_2v_3) = 3 > 2 = \gamma_2(W_5)$. Thus $b_2(W_5) = 1$. Now assume that $n \geq 6$. If we remove an edge incident to v_1, say v_1v_2, then we get $\gamma_2(W_n - v_1v_2) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$-set that contains the vertices v_1 and v_2; such set is also a 2DS of the graph $W_n - v_1v_2$. If we remove an edge non-incident to v_1, say v_2v_3, then we get $\gamma_2(W_n - v_2v_3) = \gamma_2(W_n)$ as we can construct a $\gamma_2(W_n)$-set that does not contain the vertices v_2 and v_3; such set is also a 2DS of the graph $W_n - v_2v_3$. This implies that $b_2(W_n) \neq 1$. First assume that $n = 3k$ or $n = 3k+1$. Let us remove two edges non-incident to v_1 and incident to the same vertex v_i (for some $i \neq 1$). For example, we remove the edges $v_{n-1}v_n$ and v_nv_2. Now we find a relation between the numbers $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_n - v_n)$. Let D be any $\gamma_2(W_n - v_{n-1}v_n - v_nv_2)$-set. By Observation 2 we have $v_n \in D$. Let us observe that $D \setminus \{v_n\}$ is a 2DS of the graph $W_n - v_n$. Thus $\gamma_2(W_n - v_n) \leq \gamma_2(W_n - v_{n-1}v_n - v_nv_2) - 1$. Observe that $W_n - v_n$ is a subgraph of W_{n-1} having the same set of vertices, as $W_{n-1} - v_{n-1}v_2 = W_n - v_n$. Using Observations 3 and 7 we get $\gamma_2(W_n - v_{n-1}v_n - v_nv_2) \geq \gamma_2(W_n - v_n) + 1 \geq \gamma_2(W_{n-1}) + 1 = \lfloor n/3 \rfloor + 2 = \lfloor (n+1)/3 \rfloor + 2 = \gamma_2(W_n) + 1 > \gamma_2(W_n)$. Therefore $b_2(W_n) = 2$ if $n = 3k$ or $n = 3k+1$. Now assume that $n = 3k+2$. It is not very difficult to verify that now removing any two edges does not increase the 2-domination number. This implies that $b_2(W_n) \neq 1, 2$. Let us remove three edges non-incident to v_1, and forming a path P_4. For example, we remove the edges $v_{n-2}v_{n-1}$, $v_{n-1}v_n$, and v_nv_2. Now we find a relation between the numbers $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$ and $\gamma_2(W_n - v_{n-1} - v_n)$. Let D be any $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2)$-set. By Observation 2 we have $v_{n-1}, v_n \in D$. Let us observe that $D \setminus \{v_{n-1}, v_n\}$ is a 2DS of the graph $W_n - v_{n-1} - v_n$. Thus $\gamma_2(W_n - v_{n-1} - v_n) \le \gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) - 2$. Observe that $W_n - v_{n-1} - v_n$ is a subgraph of W_{n-2} having the same set of vertices, as $W_{n-2} - v_{n-2}v_2 = W_n - v_{n-1} - v_n$. Using Observations 3 and 7 we get $\gamma_2(W_n - v_{n-2}v_{n-1} - v_{n-1}v_n - v_nv_2) \ge \gamma_2(W_n - v_{n-1} - v_n) + 2 \ge \gamma_2(W_{n-2}) + 2$ $= \lfloor (n-1)/3 \rfloor + 3 = \lfloor (3k+1)/3 \rfloor + 3 = \lfloor (3k+3)/3 \rfloor + 2 = \lfloor (n+1)/3 \rfloor + 2$ $= \gamma_2(W_n) + 1 > \gamma_2(W_n)$. Therefore $b_2(W_n) = 3$ if $n = 3k + 2$. ■

Now we investigate the 2-bondage in complete bipartite graphs.

Proposition 13 Let p and q be positive integers such that $p \le q$. Then

$$b_2(K_{p,q}) = \begin{cases} q-1 & \text{if } p = 1; \\ 3 & \text{if } p = q = 3; \\ 5 & \text{if } p = q = 4; \\ p-1 & \text{otherwise.} \end{cases}$$

Proof Let $E(K_{p,q}) = \{a_ib_j : 1 \le i \le p \text{ and } 1 \le j \le q\}$. If $p = 1$, then $K_{p,q}$ is a star. We have $b_2(K_{1,1}) = 0 = q - 1$. If $q \ge 2$, then it is not difficult to verify that in order to increase the 2-domination number we have to remove all but one edge of $K_{1,q}$. Thus $b_2(K_{1,q}) = q - 1$.

Now assume that $p = 2$. By Observation 8 we have $\gamma_2(K_{2,q}) = 2$. Let us observe that $\gamma_2(K_{2,q} - a_1b_1) = 3$. Consequently, $b_2(K_{2,q}) = 1 = p - 1$.

Now let us assume that $p = 3$. By Observation 8 we have $\gamma_2(K_{3,q}) = 3$. If $q = 3$, then it is not difficult to verify that removing any two edges does not increase the 2-domination number. We have $\gamma_2(K_{3,3} - a_1b_1 - a_1b_2 - a_2b_1) = 4 > 3$ $= \gamma_2(K_{3,3})$. Therefore $b_2(K_{3,3}) = 3$. Now assume that $q \ge 4$. It is easy to see that removing one edge does not increase the 2-domination number. Let us observe that $\gamma_2(K_{3,q} - a_1b_1 - a_2b_1) = 4$. Therefore $b_2(K_{3,q}) = 2 = p - 1$ if $q \ge 4$.

Now assume that $p \ge 4$. By Observation 8 we have $\gamma_2(K_{p,q}) = 4$. If $q = 4$, then it is not very difficult to verify that removing any four edges does not increase the 2-domination number. Let us observe that $\gamma_2(K_{4,4} - a_1b_1 - a_1b_2 - a_1b_3 - a_2b_1 - a_3b_1)$ $= 5$. Consequently, $b_2(K_{4,4}) = 5$. Now assume that $q \ge 5$. Let E' be a subset of the set of edges of $K_{p,q}$, and let $H = K_{p,q} - E'$. Let us observe that if there are vertices a_i and a_j such that $d_H(a_i) = d_H(a_j) = q$ and vertices b_k and b_l such that $d_H(b_k) = d_H(b_l) = p$, then $b_2(H) = 4$. Therefore removing any $p - 2$ edges of $K_{p,q}$ does not increase the 2-domination number. Let $E' = \{a_1b_1, a_2b_1, \dots, a_{p-1}b_1\}$. We have $\gamma_2(H) = 5$ as the vertex b_1 has to belong to every 2DS of the graph H. This implies that $b_2(K_{p,q}) = p - 1$ if $p \ge 4$ and $q \ge 5$. ■

A paired dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. The paired domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired dominating set of G. The paired bondage number, denoted by $b_p(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \ge 1$ and $\gamma_p(G - E') > \gamma_p(G)$. If for every $E' \subseteq E$, either $\gamma_p(G - E') = \gamma_p(G)$ or $\delta(G - E') = 0$, then we define $b_p(G) = 0$, and we say that G is a γ_p-strongly stable graph. Raczek [11] noticed that if $H \subseteq G$, then $b_p(H) \le b_p(G)$. Let us observe that no inequality of such type is possible for the 2-bondage. Consider the complete bipartite graphs $K_{1,3}$, $K_{2,3}$, and $K_{3,3}$. Obviously, $K_{1,3} \subseteq K_{2,3} \subseteq K_{3,3}$. Using Proposition 13 we get $b_2(K_{1,3})$

$= 2 > 1 = b_2(K_{2,3}) < 3 = b_2(K_{3,3})$.

The authors of [4] proved that the bondage number of any tree is either one or two. Let us observe that for any non-negative integer there exists a tree with such 2-bondage number, as by Proposition 13 we have $b_2(K_{1,m}) = m - 1$. Obviously, $b_2(P_1) = 0$ and $b_2(P_2) = 0$. Let us observe that the paths P_1 and P_2 are the only γ_2-strongly stable trees. We characterize all trees with 2-bondage number equaling one or two.

Let $\mathcal{T}_0$ be a family of trees that have a strong support vertex of degree three, or a vertex adjacent to at least two support vertices of degree two, or a vertex which does not belong to any minimum 2-dominating set and is adjacent to a star $K_{1,3}$ through the central vertex.

Now we prove that the 2-bondage number of every tree of the family $\mathcal{T}_0$ is either one or two.

Lemma 14 If $T \in \mathcal{T}_0$, then $b_2(T) \in \{1, 2\}$.

Proof First assume that T has a strong support vertex, say x, of degree three. Let y and z be leaves adjacent to x. The neighbor of x other than y and z is denoted by t. Let $T' = T - x - y - z$. Let D' be any $\gamma_2(T')$-set. It is easy to observe that $D' \cup \{y, z\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 2$. Now we get $\gamma_2(T - tx - xy) = \gamma_2(T' \cup P_1 \cup P_2) = \gamma_2(T') + \gamma_2(P_1) + \gamma_2(P_2) = \gamma_2(T') + 3 \geq \gamma_2(T) + 1 > \gamma_2(T)$. This implies that $0 \neq b_2(T) \leq 2$, that is, $b_2(T) \in \{1, 2\}$.

Now assume that T has a vertex, say x, adjacent to at least two support vertices of degree two. One of them let us denote by y. The leaf adjacent to y is denoted by z. Let $T' = T - y - z$. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex x. Let D' be such a set. It is easy to see that $D' \cup \{z\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now we get $\gamma_2(T - xy) = \gamma_2(T' \cup P_2) = \gamma_2(T') + \gamma_2(P_2) = \gamma_2(T') + 2 \geq \gamma_2(T) + 1 > \gamma_2(T)$. This implies that $b_2(T) = 1$.

Now assume that T has a vertex, say x, which does not belong to any $\gamma_2(T)$-set, and is adjacent to a star $K_{1,3}$ through the central vertex, say y. The leaves adjacent to y we denote by a, b, and c. Let D be any $\gamma_2(T)$-set. By Observation 2 we have $a, b, c \in D$. The vertex x does not belong to any $\gamma_2(T)$-set, thus $x, y \notin D$. Let $T' = T - a - b$. It is easy to observe $D \setminus \{a, b\}$ is not a 2DS of the tree T' as the vertex y has only one neighbor in $D \setminus \{a, b\}$. Therefore $\gamma_2(T') > \gamma_2(T) - 2$. Now we get $\gamma_2(T - ya - yb) = \gamma_2(T' \cup P_1 \cup P_1) = \gamma_2(T') + 2\gamma_2(P_1) = \gamma_2(T') + 2 > \gamma_2(T)$. This implies that $b_2(T) \in \{1, 2\}$. ∎

We characterize all trees with 2-bondage number equaling one or two. For this purpose we introduce a family $\mathcal{T}$, which consists of the path P_3, all trees of the family $\mathcal{T}_0$, and trees T_k that can be obtained as follows. Let T_1 be an element of $\mathcal{T}_0$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a star by joining the central vertex to any vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 and a non-negative number of vertices to a leaf of T_k.

Now we prove that the 2-bondage number of every tree of the family $\mathcal{T}$ is either one or two.

Lemma 15 If $T \in \mathcal{T}$, then $b_2(T) \in \{1, 2\}$.

Proof Obviously, $b_2(P_3) = 1$. If $T \in \mathcal{T}_0$, then by Lemma 14 we have $b_2(T) \in \{1, 2\}$. Now assume that $T \in \mathcal{T} \setminus (\mathcal{T}_0 \cup \{P_3\})$. We use the induction on the number k of operations performed to construct the tree T. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$

operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. The support vertex of the attached star $K_{1,m}$ is denoted by x. The vertex to which x is attached is denoted by y. Let D' be any $\gamma_2(T')$-set. It is easy to observe that the elements of the set D' together with the leaves of the the attached star form a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + m$. The assumption $b_2(T') \in \{1,2\}$ implies that there exists $E' \subseteq E(T')$ such that $|E'| \le 2$ and $\gamma_2(T' - E') > \gamma_2(T')$. By T^y (T'^y, respectively) we denote the component of $T - E'$ ($T' - E'$, respectively) which contains the vertex y. Let us observe that there exists a $\gamma_2(T^y)$-set that does not contain the vertex x. Let D^y be such a set. Observation 2 implies that all leaves of the attached star belong to the set D^y. Observe that after removing the leaves of the attached star from the set D^y we get a 2DS of the tree T'^y. Therefore $\gamma_2(T'^y) \le \gamma_2(T^y) - m$. Now we get $\gamma_2(T - E') = \gamma_2(T - E' - T^y) + \gamma_2(T^y) \ge \gamma_2(T - E' - T^y) + \gamma_2(T'^y) + m$ $= \gamma_2(T' - E' - T'^y) + \gamma_2(T'^y) + m = \gamma_2(T' - E') + m > \gamma_2(T') + m \ge \gamma_2(T)$. This implies that $0 \ne b_2(T) \le 2$, and consequently, $b_2(T) \in \{1,2\}$.

Now assume that T is obtained from T' by Operation $\mathcal{O}_2$. Assume that we attach one path P_2 and $k \ge 0$ vertices. The vertex to which are attached new vertices we denote by x. Let D' be any $\gamma_2(T')$-set. By Observation 2 we have $x \in D'$. It is easy to observe that the elements of the set D' together with all leaves of T which do not exist in T' form a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + k + 1$. The assumption $b_2(T') \in \{1,2\}$ implies that there exists $E' \subseteq E(T')$ such that $|E'| \le 2$ and $\gamma_2(T' - E') > \gamma_2(T')$. By T^x (T'^x, respectively) we denote the component of $T - E'$ ($T' - E'$, respectively) which contains the vertex x. Let us observe that there exists a $\gamma_2(T^x)$-set that contains the vertex x. Let D^x be such a set. Observation 2 implies that all leaves of T which do not exist in T' belong to the set D^x. The set D^x is minimal, thus no vertex of T, which neither exists in the tree T' nor is a leaf, belongs to the set D^x. It is easy to observe that after removing from D all leaves of T which do not exist in T' we get a 2DS of the tree T'^x. Therefore $\gamma_2(T'^x) \le \gamma_2(T^x) - k - 1$. Now we get $\gamma_2(T - E') = \gamma_2(T - E' - T^x) + \gamma_2(T^x)$ $\ge \gamma_2(T - E' - T^x) + \gamma_2(T'^x) + k + 1 = \gamma_2(T' - E' - T'^x) + \gamma_2(T'^x) + k + 1$ $= \gamma_2(T' - E') + k + 1 > \gamma_2(T') + k + 1 \ge \gamma_2(T)$. This implies that $b_2(T) \in \{1,2\}$. ■

Now we prove that if the 2-bondage number of a tree equals one or two, then the tree belongs to the family $\mathcal{T}$.

Lemma 16 Let T be a tree. If $b_2(T) \in \{1,2\}$, then $T \in \mathcal{T}$.

Proof Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) \in \{0,1\}$, then $T \in \{P_1, P_2\}$. We have $b_2(P_1) = b_2(P_2)$ $= 0 \notin \{1,2\}$. Now assume that $\mathrm{diam}(T) = 2$. Thus T is a star $K_{1,m}$. By Proposition 13 we have $b_2(K_{1,m}) = m - 1$. If $b_2(K_{1,m}) = 1$, then $m = 2$. We have $T = K_{1,2}$ $= P_3 \in \mathcal{T}$. If $b_2(K_{1,m}) = 2$, then $m = 3$. We have $T = K_{1,3} \in \mathcal{T}_0 \subseteq \mathcal{T}$ as $K_{1,3}$ has a strong support vertex of degree three.

Now assume that $\mathrm{diam}(T) \ge 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$. We root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \ge 4$, then let w be the parent of u. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(v) \ge 5$. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. Observation 2 implies that all leaves adjacent to v belong to the set D. Observe that after removing them from the set D we get a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T)$

$-d_T(v)+1$. The assumption $b_2(T) \in \{1,2\}$ implies that there exists $E' \subseteq E(T)$ such that $|E'| = b_2(T) \leq 2$ and $\gamma_2(T-E') > \gamma_2(T)$. In every $\gamma_2(T)$-set the vertex v has at least four neighbors. This implies that the set E' does not contain any edge incident to v. By T^u (T'^u, respectively) we denote the component of $T-E'$ ($T'-E'$, respectively) which contains the vertex u. Let D'^u be any $\gamma_2(T'^u)$-set. It is easy to observe that the elements of the set D'^u together with the leaves adjacent to v form a 2DS of the tree T^u. Thus $\gamma_2(T^u) \leq \gamma_2(T'^u) + d_T(v) - 1$. Now we get $\gamma_2(T'-E') = \gamma_2(T'-E'-T'^u) + \gamma_2(T'^u) \geq \gamma_2(T'-E'-T'^u) + \gamma_2(T^u) - d_T(v) + 1$ $= \gamma_2(T-E'-T^u) + \gamma_2(T^u) - d_T(v) + 1 = \gamma_2(T-E') - d_T(v) + 1 > \gamma_2(T) - d_T(v) + 1$ $\geq \gamma_2(T')$. This implies that $0 \neq b_2(T') \leq |E'| \leq 2$, and consequently, $b_2(T') \in \{1,2\}$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(v) = 4$. The leaves adjacent to v and different from t are denoted by a and b. If no $\gamma_2(T)$-set contains the vertex u, then $T \in \mathcal{T}_0$ as u is adjacent to a star $K_{1,3}$ through the central vertex. Now assume that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. By Observation 2 we have $t, a, b \in D$. The set D is minimal, and thus $v \notin D$. Let $T' = T - T_v$. Observe that $D \setminus \{t, a, b\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 3$. The assumption $b_2(T) \in \{1,2\}$ implies that there exists $E' \subseteq E(T)$ such that $|E'|$ $= b_2(T) \leq 2$ and $\gamma_2(T-E') > \gamma_2(T)$. The vertex v has four neighbors in D, and thus the set E' does not contain any edge incident to v. By T^u (T'^u, respectively) we denote the component of $T-E'$ ($T'-E'$, respectively) which contains the vertex u. Let D'^u be any $\gamma_2(T'^u)$-set. It is easy to observe that $D'^u \cup \{t, a, b\}$ is a 2DS of the tree T^u. Thus $\gamma_2(T^u) \leq \gamma_2(T'^u) + 3$. Now we get $\gamma_2(T'-E') = \gamma_2(T'-E'$ $-T'^u) + \gamma_2(T'^u) \geq \gamma_2(T'-E'-T'^u) + \gamma_2(T^u) - 3 = \gamma_2(T-E'-T^u) + \gamma_2(T^u) - 3$ $= \gamma_2(T-E') - 3 > \gamma_2(T) - 3 \geq \gamma_2(T')$. Now we conclude that $b_2(T') \in \{1,2\}$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(v) = 3$. The vertex v is a strong support vertex of degree three. Thus $T \in \mathcal{T}_0 \subseteq \mathcal{T}$.

Now assume that $d_T(v) = 2$. First assume that some child of u other than v, say x, is a support vertex. It suffices to consider only the possibility when x is adjacent to exactly one leaf. The vertex u is adjacent to at least two support vertices of degree two. Thus $T \in \mathcal{T}_0 \subseteq \mathcal{T}$.

Now assume that every child of u different from v is a leaf. Let T' be a tree that differs from $T - T_u$ only in that it has the vertex u. Let us observe that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. Observation 2 implies that all leaves of T_u belong to the set D. Since D is minimal, it does not contain any vertex, which neither exists in the tree T' nor is a leaf. It is easy to observe that after removing from D all leaves of T_u we get a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - d_T(u) + 1$. The assumption $b_2(T) \in \{1,2\}$ implies that there exists $E' \subseteq E(T)$ such that $|E'| = b_2(T) \leq 2$ and $\gamma_2(T-E') > \gamma_2(T)$. Let us observe that the set E' does not contain any edge incident to a leaf adjacent to u. Assume that E' contains uv or vt. This implies that no $\gamma_2(T)$-set contains the vertex v. Let us observe that $\gamma_2(T'-wu) > \gamma_2(T')$. This implies that $b_2(T') = 1$. Now assume that the set E' does not contain any edge of T_u. By T^u (T'^u, respectively) we denote the component of $T-E'$ ($T'-E'$, respectively) which contains the vertex u. Let D'^u be any $\gamma_2(T'^u)$-set. By Observation 2 we have $u \in D'^u$. It is easy to observe that the elements of the set D'^u together with all leaves of T_u form a 2DS of the tree T^u. Thus $\gamma_2(T^u) \leq \gamma_2(T'^u) + d_T(u) - 1$. Now we get $\gamma_2(T'-E') = \gamma_2(T'-E'-T'^u) + \gamma_2(T'^u)$ $\geq \gamma_2(T'-E'-T'^u) + \gamma_2(T^u) - d_T(u) + 1 = \gamma_2(T-E'-T^u) + \gamma_2(T^u) - d_T(u) + 1$ $= \gamma_2(T-E') - d_T(u) + 1 > \gamma_2(T) - d_T(u) + 1 \geq \gamma_2(T')$. Now we conclude that

$b_2(T') \in \{1,2\}$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. ■

As an immediate consequence of Lemmas 15 and 16, we have the following characterization of trees with 2-bondage number equaling one or two.

Theorem 2.1 Let T be a tree. Then $b_2(T) \in \{1,2\}$ if and only if $T \in \mathcal{T}$.

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