ON TREES WITH DOUBLE DOMINATION NUMBER EQUAL TO 2-DOMINATION NUMBER PLUS ONE

MARCIN KRZYWKOWSKI

Abstract. A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a 2-dominating set of G if every vertex of $V(G) \setminus D$ is dominated by at least two vertices of D, while it is a double dominating set of G if every vertex of G is dominated by at least two vertices of D. The 2-domination (double domination, respectively) number of a graph G is the minimum cardinality of a 2-dominating (double dominating, respectively) set of G. We characterize all trees with the double domination number equal to the 2-domination number plus one.

1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a subset of $V(G)$ is independent if there is no edge between every two its vertices. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges, ux and vx. By a subdivided star we mean a graph obtained from a star by subdividing each one of its edges.

2000 *Mathematics Subject Classification.* 05C05, 05C69.

Key words and phrases. 2-domination, double domination, tree.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set, abbreviated 2DS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. A 2-dominating set of G of minimum cardinality is called a $\gamma_2(G)$-set. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [5], and further studied for example in [3, 6, 11, 12]. For a comprehensive survey of domination in graphs, see [9, 10].

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a double dominating set, abbreviated DDS, of G if every vertex of G is dominated by at least two vertices of D. The double domination number of G, denoted by $\gamma_d(G)$, is the minimum cardinality of a double dominating set of G. A double dominating set of G of minimum cardinality is called a $\gamma_d(G)$-set. Double domination in graphs was introduced by Harary and Haynes [8], and further studied for example in [1, 4, 7].

It is not difficult to observe that every double dominating set of a graph G is a 2-dominating set of this graph. Thus $\gamma_d(G) \geq \gamma_2(G)$, for every graph G.

A paired dominating set of a graph is a dominating set of vertices whose induced subgraph has a perfect matching. The authors of [2] characterized all trees with equal double domination and paired domination numbers.

We characterize all trees with the double domination number equal to the 2-domination number plus one.

2. Results

Since the one-vertex graph does not have double dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following three straightforward observations.

Observation 2.1. *Every leaf of a graph G is in every $\gamma_2(G)$-set.*

Observation 2.2. *Every leaf of a graph G is in every $\gamma_d(G)$-set.*

Observation 2.3. *Every support vertex of a graph G is in every $\gamma_d(G)$-set.*

It is easy to see that $\gamma_d(P_2) = \gamma_2(P_2) = 2$. Now we prove that for every tree different than P_2, the double domination number is greater than the 2-domination number.

Lemma 2.4. *For every tree $T \neq P_2$ we have $\gamma_d(T) > \gamma_2(T)$.*

Proof. Since $T \neq P_2$, we have $\text{diam}(T) \geq 2$. If $\text{diam}(T) = 2$, then T is a star $K_{1,m}$. We have $\gamma_d(T) = m + 1 > m = \gamma_2(T)$. Now let us assume that $\text{diam}(T) = 3$. Thus T is a double star. Let n mean the order of the tree T. We have $\gamma_d(T) = n > n - 1 \geq \gamma_2(T)$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z be leaves adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_2(T')$-set. Of course, $D' \cup \{y\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $y, z, x \in D$. It is easy to see that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 1 > \gamma_2(T') + 1 \geq \gamma_2(T)$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that u is adjacent to a leaf, say x. Let $T' = T - T_v$. Let D' be any $\gamma_2(T')$-set. Of course, $D' \cup \{v, t\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 2$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t, x, v, u \in D$. It is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 2 > \gamma_2(T') + 2 \geq \gamma_2(T)$.

Now assume that among the descendants of u there is a support vertex, say x, different than v. The leaf adjacent to x we denote by y. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. If $u \in D$, then it is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Now assume that $u \notin D$. Let us observe that $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T) \geq \gamma_d(T') + 1 > \gamma_2(T') + 1 \geq \gamma_2(T)$.

Now assume that $d_T(u) = 2$. Let $T' = T - T_v$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $u \in D'$. It is easy to see that $D' \cup \{t\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t, v \in D$. Let us observe that both vertices u and w cannot at the same time be outside D as the vertex u has to be dominated at least twice. If $u, w \in D$, then it is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. If $u \in D$ and $w \notin D$, then it is easy to observe that $D \cup \{w\} \setminus \{v, t\}$ is a DDS of the tree T'. Similarly, if $w \in D$ and $u \notin D$, then $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Now we conclude that $\gamma_d(T') \le \gamma_d(T) - 1$. We get $\gamma_d(T) \ge \gamma_d(T') + 1 > \gamma_2(T') + 1 \ge \gamma_2(T)$. □

Now we give a necessary condition for that the double domination number of a tree is equal to its 2-domination number plus one.

Lemma 2.5. *If $\gamma_d(T) = \gamma_2(T) + 1$, then for every $\gamma_d(T)$-set D, every vertex of $V(T) \setminus D$ has degree two.*

Proof. Suppose that there exists a $\gamma_d(T)$-set D that does not contain a vertex of T, say x, which has degree different than two. By Observation 2.2, every leaf belongs to the set D. Therefore $d_T(x) \ge 3$. First assume that some neighbor of x, say y, also does not belong to the set D. By T_1 and T_2 we denote the trees resulting from T by removing the edge xy. Let us observe that each one of those trees has at least three vertices. We define $D_1 = D \cap V(T_1)$ and $D_2 = D \cap V(T_2)$. Let us observe that D_1 is a DDS of the tree T_1 and D_2 is a DDS of the tree T_2. Let D'_1 be any $\gamma_2(T_1)$-set and let D'_2 be any $\gamma_2(T_2)$-set. By Lemma 2.4 we have $\gamma_d(T_1) \ge \gamma_2(T_1) + 1$ and $\gamma_d(T_2) \ge \gamma_2(T_2) + 1$. Of course, $D'_1 \cup D'_2$ is a 2DS of the tree T. Thus $\gamma_2(T) \le |D'_1 \cup D'_2|$. Now we get $\gamma_d(T) = |D| = |D_1 \cup D_2|$ $= |D_1| + |D_2| \ge \gamma_d(T_1) + \gamma_d(T_2) \ge \gamma_2(T_1) + 1 + \gamma_2(T_2) + 1 = |D'_1| + |D'_2| + 2$ $= |D'_1 \cup D'_2| + 2 \ge \gamma_2(T) + 2 > \gamma_2(T) + 1$, a contradiction.

Now assume that all neighbors of x belong to the set D. First assume that there is a neighbor of x, say y, such that each one of the two trees resulting from T by removing the edge xy has at least three vertices. We get a contradiction similarly as when some neighbor of x does not belong to the set D. Now assume that there is no neighbor of x such that each one of the two trees resulting from T by removing the edge between them has at least three vertices. This implies that T is a subdivided star of order at least seven. Let n mean the number of vertices of the tree T. We have $\gamma_d(T) = n - 1 = (n + 1)/2 + 1 + (n - 5)/2$ $= \gamma_2(T) + 1 + (n - 5)/2 > \gamma_2(T) + 1$, a contradiction. □

Let T be a tree. If T is a path, then let $C(T)$ be a one-element set containing a support vertex of T. If T is not a path, then let $C(T)$ be a set of vertices of T which have degree at least three. We say that two vertices of $C(T)$ are linked if the path joining them in T is such that all its interior vertices have degree two. Then the path is called a link. The length of a link is the number of its edges. Paths joining leaves of T to vertices of $C(T)$ we call chains. The length of a chain is the number of its edges.

Let $\mathcal{T}_0$ be a family of trees T such that every link has length two, every chain has length one or three, and each vertex of $C(T)$ is adjacent to at least one chain of length one.

Now we prove that for every tree of the family $\mathcal{T}_0$, the double domination number is equal to the 2-domination number plus one.

Lemma 2.6. *If* $T \in \mathcal{T}_0$*, then* $\gamma_d(T) = \gamma_2(T) + 1$.

Proof. Let us observe that for any tree T the following algorithm finds a 2-dominating set of minimum cardinality. Label vertices of T as taken, omitted, and undecided. Initialize by calling every vertex undecided. Root T at any vertex, say r. Let $v \neq r$ be a vertex of T, which has not already been decided, and such that all its children have been decided. If all children of v have been omitted, then take v. If exactly one child of v has been taken, then omit v and take its parent. If at least two children of v have been taken, then omit v. When all children of r are decided, take r if at most one child of r has been taken; otherwise omit r. It is not very difficult to observe that the taken vertices form a $\gamma_2(T)$-set.

By Observations 2.2 and 2.3, every DDS of T contains all leaves and support vertices. Let us observe that the set of all leaves and support vertices is a DDS of the tree T. Therefore these vertices form a $\gamma_d(T)$-set. Rooting T at the center vertex of a link, and running the algorithm above we see that a $\gamma_2(T)$-set contains all vertices of T excluding support vertices. Let us observe that the number of non-support vertices of T is one less than the number of all leaves and support vertices of T. Therefore $\gamma_d(T) = \gamma_2(T) + 1$. □

We characterize all trees with the double domination number equal to the 2-domination number plus one. For this purpose we introduce a family $\mathcal{T}$ of trees T that either belong to the family $\mathcal{T}_0$, or can be obtained from an element of $\mathcal{T}_0$, say T', in the following way. Let x mean a leaf of T'. If the neighbor of x is a strong support vertex or has degree at least three, then we can attach a vertex by joining it to the leaf x. If the neighbor of x is a strong support vertex, then we can attach a tree of the family $\mathcal{T}_0$ by joining its any leaf to the leaf x.

Now we prove that for every tree of the family $\mathcal{T}$, the double domination number is equal to the 2-domination number plus one.

Lemma 2.7. *If* $T \in \mathcal{T}$*, then* $\gamma_d(T) = \gamma_2(T) + 1$.

Proof. If $T \in \mathcal{T}_0$, then by Lemma 2.6 we have $\gamma_d(T) = \gamma_2(T) + 1$. Now assume that $T \in \mathcal{T} \setminus \mathcal{T}_0$. First assume that T can be obtained from an element of $\mathcal{T}_0$, say T', by attaching a vertex, say w, by joining it to a leaf of T', say x. The neighbor of x we denote by y. The vertex y is a strong support vertex or has degree at least three. Let D' be any $\gamma_d(T')$-set. By Observation 2.2 we have $x \in D'$. It is easy to see that $D' \cup \{w\}$ is a DDS of the tree T. Thus $\gamma_d(T) \le \gamma_d(T') + 1$. Rooting T at the vertex x, and running the earlier algorithm we get a $\gamma_2(T)$-set which contains the vertex x. Let D be such a set. By Observation 2.1 we have $w \in D$. It is easy to see that $D \setminus \{w\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get $\gamma_d(T) \le \gamma_d(T') + 1 = \gamma_2(T') + 2 \le \gamma_2(T) + 1$. On the other hand, by Lemma 2.4 we have $\gamma_d(T) \ge \gamma_2(T) + 1$. This implies that $\gamma_d(T) = \gamma_2(T) + 1$.

Now assume that T can be obtained from an element of $\mathcal{T}_0$, say T', by attaching a tree of the family $\mathcal{T}_0$, say H, by joining its leaf, say w, to a leaf of T', say x, adjacent to a strong support vertex, say y. Let z mean a leaf adjacent to y and different from x. Let D' be any $\gamma_d(T')$-set and let D_H be any $\gamma_d(H)$-set. By Observations 2.2 and 2.3 we have $x, y, z \in D'$ and $w \in D_H$. It is easy to observe that $D' \cup D_H \setminus \{x\}$ is a DDS of the tree T. Thus $\gamma_d(T) \le \gamma_d(T') + \gamma_d(H) - 1$. Rooting T at the vertex x, and running the earlier algorithm we get a $\gamma_2(T)$-set that contains the vertices x and w. Let D be such a set. It is easy to see that $D \cap V(T')$ is a 2DS of the tree T' and $D \cap V(H)$ is a 2DS of the tree H. Therefore $\gamma_2(T') + \gamma_2(H) \le \gamma_2(T)$. Now we get $\gamma_d(T) \le \gamma_d(T') + \gamma_d(H) - 1$ $= \gamma_2(T') + 1 + \gamma_2(H) + 1 - 1 = \gamma_2(T') + \gamma_2(H) + 1 \le \gamma_2(T) + 1$. This implies that $\gamma_d(T) = \gamma_2(T) + 1$. □

Now we prove that if the double domination number of a tree is equal to its 2-domination number plus one, then the tree belongs to the family $\mathcal{T}$.

Lemma 2.8. *Let T be a tree. If $\gamma_d(T) = \gamma_2(T) + 1$, then $T \in \mathcal{T}$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\operatorname{diam}(T) = 1$, then $T = P_2$. We have $\gamma_d(T) = 2 = \gamma_2(T)$ $\neq \gamma_2(T) + 1$. If $\operatorname{diam}(T) = 2$, then T is a star. It is easy to see that $T \in \mathcal{T}_0 \subseteq \mathcal{T}$.

Now assume that $\operatorname{diam}(T) \ge 3$. Thus the order of the tree T is an integer $n \ge 4$. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that T has a chain of length at least seven, say ending $gfedcba$, where a is a leaf. Let $T' = T - a - b - c - d - e - f$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $g \in D'$. It is easy to observe that $D' \cup \{e, c, a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices c and f. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b \in D$. The vertex d has to be dominated twice, thus $d, e \in D$. Observe that $D \setminus \{e, d, b, a\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 4$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 4 = \gamma_2(T) - 3 \leq \gamma_2(T')$. This is a contradiction as by Lemma 2.4 we have $\gamma_d(T') > \gamma_2(T')$. Therefore every chain of T has length at most six.

Now assume that some vertex of $C(T)$, say x, is adjacent to a chain of length six, say $xfedcba$. Let $T' = T - a - b - c$ and $T'' = T' - d$. Let D' be any $\gamma_2(T')$-set. It is easy to see that $D' \cup \{a, c\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex c. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b \in D$. Observe that $D \setminus \{a, b\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_2(T) - 1 \leq \gamma_2(T') + 1$. On the other hand, by Lemma 2.4 we have $\gamma_d(T') \geq \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Let D'' be any $\gamma_2(T'')$-set. By Observation 2.1 we have $e \in D''$. It is easy to observe that $D'' \cup \{c, a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T'') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices c and f. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b \in D$. The vertex c has to be dominated twice, thus $d \in D$. Let us observe that $D \cup \{f\} \setminus \{d, b, a\}$ is a DDS of the tree T''. Therefore $\gamma_d(T'') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T'') \leq \gamma_d(T) - 2 = \gamma_2(T) - 1 \leq \gamma_2(T'') + 1$. This implies that $\gamma_d(T'') = \gamma_2(T'') + 1$. By the inductive hypothesis we have $T'' \in \mathcal{T}$. Observe that $T'' \notin \mathcal{T}_0$ as the tree T'' has a chain of length two. Thus $T'' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that tree T'' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let $T''' = T'' - d$. Let us observe that the only components which can form the tree T'' are T''' and the one-vertex graph. Thus $T''' \in \mathcal{T}_0$. Since $T' \in \mathcal{T}$, it follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that x is a strong support vertex of T'''. The tree T can be obtained from T''' by attaching a path P_5 by joining its any leaf to the leaf f. Thus $T \in \mathcal{T}$.

Now assume that some vertex of $C(T)$, say x, is adjacent to a chain of length five, say $xedcba$. Let $T' = T - a - b - c - d$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $e \in D'$. It is easy to observe that $D' \cup \{c, a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex c. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b \in D$. The vertex d has to be dominated twice, thus $d, e \in D$. By Lemma 2.5 we have $x \in D$. It is easy to see that $D \setminus \{d, b, a\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 3$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 3 = \gamma_2(T) - 2 \leq \gamma_2(T')$, a contradiction.

Now assume that some vertex of $C(T)$, say x, is adjacent to a chain of length four, say $xdcba$. Let $T' = T - a - b$ and $T'' = T' - c$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $c \in D'$. It is easy to see that $D' \cup \{a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex c. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b \in D$. Let us observe that $D \cup \{c\} \setminus \{a, b\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2(T) - 1 \le \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Observe that $T' \notin \mathcal{T}_0$ as the tree T' has a chain of length two. Thus $T' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that the tree T' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let us observe that the only components which can form the tree T' are T'' and the one-vertex graph. Thus $T'' \in \mathcal{T}_0$. The tree T can be obtained from T'' by attaching a path P_3 by joining its any leaf to the leaf d. Thus $T \in \mathcal{T}$.

Now assume that every chain of T has length at most three. First assume that the set $C(T)$ contains exactly one vertex, say x. Thus the tree T can be obtained from a star by subdividing each one of its edges at most twice. Assume that x is adjacent to at least two chains of length two. Let xba and xdc mean chains adjacent to x. Let $T' = T - a - b$. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex x. Let D' be such a set. It is easy to see that $D' \cup \{a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $a, b, d \in D$. By Lemma 2.5 we have $x \in D$. It is easy to see that $D \setminus \{a, b\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2(T) - 1 \le \gamma_2(T')$, a contradiction. Therefore x is adjacent to at most one chain of length two. If x is adjacent to a chain of length one or two, then from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ it follows that $T \in \mathcal{T}$. Now assume that x is not adjacent to any chain of length one or two. Thus every chain adjacent to x has length three. We have $\gamma_d(T) = n - d_T(x) + 1 = n - d_T(x) - 1 + 2 = \gamma_2(T) + 2 > \gamma_2(T) + 1$, a contradiction.

Now assume that the set $C(T)$ has at least two elements. Let x mean a vertex of $C(T)$ adjacent to exactly one link. Thus x is adjacent to at least two chains. First assume that x is adjacent to a chain of length three, say $xcba$. Assume that $d_T(x) \ge 4$. Let $T' = T - a - b - c$. Let D' be any $\gamma_2(T')$-set. It is easy to see that $D' \cup \{a, c\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex c. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b \in D$. Observe that $D \setminus \{a, b\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get

$\gamma_d(T') \le \gamma_d(T)-2 = \gamma_2(T)-1 \le \gamma_2(T')+1$. This implies that $\gamma_d(T') = \gamma_2(T')+1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that $d_T(x) = 3$. First assume that the chain adjacent to x and different from $xcba$ has length three. Let $xfed$ mean this chain. The neighbor of x other than c and f we denote by y. First assume that $d_T(y) \ge 3$. Let $T' = T - a - b$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $c \in D'$. It is easy to see that $D' \cup \{a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $a, b \in D$. By Lemma 2.5 we have $x, y \in D$. The set D is minimal, thus $c \notin D$. Observe that $D \setminus \{a, b\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T') \le \gamma_d(T) - 1 = \gamma_2(T) \le \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. This is a contradiction as no tree of the family $\mathcal{T}$ has a link of length one.

Now assume that $d_T(y) = 2$. The neighbor of y other than x we denote by z. First assume that $d_T(z) \ge 3$. Let $T' = T - a - b - c - d - e - f - x$. Let D' be any $\gamma_2(T')$-set. It is easy to observe that $D' \cup \{a, c, d, f\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices c and f. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, d, e \in D$. By Lemma 2.5 we have $x, z \in D$. The vertex x has to be dominated twice, thus $y \in D$. It is easy to see that $D \setminus \{a, b, d, e, x\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 5$. Now we get $\gamma_d(T') \le \gamma_d(T) - 5 = \gamma_2(T) - 4 \le \gamma_2(T')$, a contradiction.

Now assume that $d_T(z) = 2$. The neighbor of z other than y we denote by k. First assume that $d_T(k) \ge 3$. Let $T' = T - a - b$ and $T'' = T' - c - d - e - f$. By T_x (T_k, respectively) we denote the component of $T - yz$ which contains the vertex x (k, respectively). Let T'_x mean the component of $T' - yz$ which contains the vertex x. Similarly as earlier we conclude that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Observe that $T' \notin \mathcal{T}_0$ as the tree T' has a link of length three. Thus $T' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that the tree T' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let us observe the only components which can form the tree T' are T'_x and T_k. Thus $T_k \in \mathcal{T}_0$. Let D'' be any $\gamma_2(T'')$-set. It is easy to observe that $D'' \cup \{a, c, d, f\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T'') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices c and f. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, d, e \in D$. Observe that $D \setminus \{a, b, d, e\}$ is a DDS of the tree T''. Therefore $\gamma_d(T'') \le \gamma_d(T) - 4$. Now we get $\gamma_d(T'') \le \gamma_d(T) - 4 = \gamma_2(T) - 3 \le \gamma_2(T'') + 1$. This implies that $\gamma_d(T'') = \gamma_2(T'') + 1$.

By the inductive hypothesis we have $T'' \in \mathcal{T}$. Since $T' \in \mathcal{T}_0$, it follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T'' \in \mathcal{T}_0$. Thus k is adjacent to a leaf in T'', and consequently, the vertex k is a strong support vertex of T_k. The tree T can be obtained from the trees T_x and T_k by joining the leaves y and z. Thus $T \in \mathcal{T}$.

Now assume that $d_T(k) = 2$. Let $T' = T - a - b - c - d - e - f - x - y - z$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $k \in D'$. It is easy to observe that $D' \cup \{y, a, c, d, f\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 5$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices c, f, and z. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, d, e \in D$. The vertex x has to be dominated twice, thus $x, y \in D$. Observe that $D \setminus \{a, b, d, e, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 6$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 6 = \gamma_2(T) - 5 \leq \gamma_2(T')$, a contradiction.

Now assume that the chain adjacent to x and different from $xcba$ has length two. Let xed mean this link. The neighbor of x other than c and e we denote by y. Let $T' = T - a - b$. Similarly as earlier we conclude that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Observe that $T' \notin \mathcal{T}_0$ as the tree T' has a chain of length two. Thus $T' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that the tree T' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let us observe that the only components which can form the tree T' are $T' - d$ and the one-vertex graph. Thus $T' - d \in \mathcal{T}_0$. Let $T'' = T - d$. It follows from the definition of the family $\mathcal{T}_0$ that $T'' \in \mathcal{T}_0$. The tree T can be obtained from T'' by attaching a vertex by joining it to the leaf c. Thus $T \in \mathcal{T}$.

Now assume that the chain adjacent to x and different from $xcba$ has length one. Let $T' = T - a - b$. Similarly as earlier we conclude that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that every chain adjacent to x has length at most two. First assume that x is adjacent to a chain of length two, say xba. Assume that x is also adjacent to another chain of length two, say xdc. Let $T' = T - a - b$. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex x. Let D' be such a set. It is easy to see that $D' \cup \{a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $a, b, d \in D$. By Lemma 2.5 we have $x \in D$. It is easy to see that $D \setminus \{a, b\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_2(T) - 1 \leq \gamma_2(T')$, a contradiction.

Thus every chain adjacent to x and different from xba has length one. Let xc mean a chain adjacent to x. First assume that $d_T(x) \geq 4$. Let $T' = T - c$.

Let D' be any $\gamma_2(T')$-set. Of course, $D' \cup \{c\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $b, c \in D$. By Lemma 2.5 we have $x \in D$. It is easy to see that $D \setminus \{c\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T') \le \gamma_d(T) - 1 = \gamma_2(T) \le \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that $d_T(x) = 3$. The neighbor of x other than b and c we denote by y. First assume that $d_T(y) \ge 3$. Let $T' = T - a$. Let D' be any $\gamma_2(T')$-set. Of course, $D' \cup \{a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $a, b, x \in D$. It is easy to see that $D \setminus \{a\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T') \le \gamma_d(T) - 1 = \gamma_2(T) \le \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. This is a contradiction as no tree of the family $\mathcal{T}$ has a link of length one.

Now assume that $d_T(y) = 2$. The neighbor of y other than x we denote by z. First assume that $d_T(z) \ge 3$. Let T' be a tree obtained from $T - a - b - c$ by attaching a vertex, say t, by joining it to the vertex z. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex z. Let D' be such a set. By Observation 2.1 we have $x, t \in D'$. It is easy to observe that $D' \setminus \{t\} \cup \{a, c\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex y. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, c, x \in D$. By Lemma 2.5 we have $z \in D$. It is easy to observe that $D \cup \{t, y\} \setminus \{a, b, c\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T') \le \gamma_d(T) - 1 = \gamma_2(T) \le \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Observe that $T' \notin \mathcal{T}_0$ as the tree T' has a chain of length two. Thus $T' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that the tree T' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let us observe that the only components which can form the tree T' are $T' - x$ and the one-vertex graph. Thus $T' - x \in \mathcal{T}_0$. Let $T'' = T - a$. It follows from the definition of the family $\mathcal{T}_0$ that $T'' \in \mathcal{T}_0$. The tree T can be obtained from T'' by attaching a vertex by joining it to the leaf b. Thus $T \in \mathcal{T}$.

Now assume that $d_T(z) = 2$. Let $T' = T - a - b - c - x - y$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $z \in D'$. It is easy to observe that $D' \cup \{a, x, c\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex y. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, c, x \in D$. Observe that

$D \setminus \{a, b, c, x\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 4$. Now we get $\gamma_d(T') \le \gamma_d(T) - 4 = \gamma_2(T) - 3 \le \gamma_2(T')$, a contradiction.

Now assume that every chain adjacent to x has length one. Let xa and xb mean chains adjacent to x. First assume that $d_T(x) \ge 4$. Let $T' = T - a$. Let D' be any $\gamma_2(T')$-set. Of course, $D' \cup \{a\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $a, b, x \in D$. It is easy to see that $D \setminus \{a\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T') \le \gamma_d(T) - 1 = \gamma_2(T) \le \gamma_2(T') + 1$. This implies that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that $d_T(x) = 3$. The neighbor of x other than a and b we denote by y. First assume that $d_T(y) \ge 3$. Let u mean a vertex of $C(T)$ other than x and adjacent to exactly one link. It suffices to consider only the possibility when $d_T(u) = 3$ and both chains adjacent to u have length one. First assume that $u \ne y$. Let ut mean a chain adjacent to u. Let $T' = T - t$. Similarly as earlier we conclude that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. This is a contradiction as no tree of the family $\mathcal{T}$ has a link of length one. Thus $u = y$. This implies that T is a double star with both support vertices of degree three. We have $\gamma_d(T) = 6 = 4 + 2 = \gamma_2(T) + 2 > \gamma_2(T) + 1$, a contradiction.

Now assume that $d_T(y) = 2$. The neighbor of y other than x we denote by z. First assume that $d_T(z) \ge 3$. Let $T' = T - a$. Similarly as earlier we conclude that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that $d_T(z) = 2$. The neighbor of z other than y we denote by k. First assume that $d_T(k) \ge 3$. Let $T' = T - a$ and $T'' = T' - b - x - y$. Similarly as earlier we conclude that $T' \in \mathcal{T}$. Observe that $T' \notin \mathcal{T}_0$ as the tree T' has a chain of length four. Thus $T' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that the tree T' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let us observe that the only components which can form the tree T' are T'' and P_3. Thus $T'' \in \mathcal{T}_0$. It is easy to see that $K_{1,3} \in \mathcal{T}_0$. The tree T can be obtained from T'' by attaching a star $K_{1,3}$ by joining its any leaf to the leaf z. Thus $T \in \mathcal{T}$.

Now assume that $d_T(k) = 2$. The neighbor of k other than z we denote by l. First assume that $d_T(l) \ge 3$. Let $T' = T - a - b - x - y - z$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $k \in D'$. It is easy to observe that $D' \cup \{y, a, b\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex y. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, x \in D$. The vertex z has to be dominated twice, thus $z, k \in D$. By Lemma 2.5 we have $l \in D$. It is easy to see that

$D \setminus \{a, b, x, z\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 4$. Now we get $\gamma_d(T') \le \gamma_d(T) - 4 = \gamma_2(T) - 3 \le \gamma_2(T')$, a contradiction.

Now assume that $d_T(l) = 2$. The neighbor of l other than k we denote by m. First assume that $d_T(m) \ge 3$. Let $T' = T - a$. Similarly as earlier we conclude that $\gamma_d(T') = \gamma_2(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Observe that $T' \notin \mathcal{T}_0$ as the tree T' has a chain of length six. Thus $T' \in \mathcal{T} \setminus \mathcal{T}_0$. This implies that the tree T' can be obtained in a way described in the definition of the family $\mathcal{T}$. Let $T'' = T' - b - x - y$ and $T''' = T'' - z - k$. Let us observe that the only two possibilities of the components which can form the tree T' are T'' with P_3 and T''' with P_5. If T is obtained from T'' by attaching a path P_3 by joining its any leaf to the leaf z, then $T'' \in \mathcal{T}_0$. Thus m is adjacent to a leaf in T''. It follows from the definitions of the families $\mathcal{T}_0$ and $\mathcal{T}$ that $T''' \in \mathcal{T}_0$. Moreover, the vertex m is a strong support vertex of the tree T'''. Now assume that T is obtained from T''' by attaching a path P_5 by joining its any leaf to the leaf l. Thus $T''' \in \mathcal{T}_0$. Moreover, the vertex m is a strong support vertex. Let $T_x = T - T'''$. The tree T can be obtained from T''' and T_x by joining the leaves k and l. Thus $T \in \mathcal{T}$.

Now assume that $d_T(m) = 2$. Let $T' = T - a - b - x - y - z - k - l$. Let D' be any $\gamma_2(T')$-set. By Observation 2.1 we have $m \in D'$. It is easy to observe that $D' \cup \{k, y, a, b\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \le \gamma_2(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices y and l. Let D be such a set. By Observations 2.2 and 2.3 we have $a, b, x \in D$. The vertex z has to be dominated twice, thus $z, k \in D$. Observe that $D \setminus \{a, b, x, z, k\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 5$. Now we get $\gamma_d(T') \le \gamma_d(T) - 5 = \gamma_2(T) - 4 \le \gamma_2(T')$, a contradiction. □

As an immediate consequence of Lemmas 2.7 and 2.8, we have the following characterization of the trees with double domination number equal to 2-domination number plus one.

Theorem 2.9. *Let T be a tree. Then $\gamma_d(T) = \gamma_2(T) + 1$ if and only if $T \in \mathcal{T}$.*

Acknowledgments. Thanks are due to Mustapha Chellali for suggesting Lemma 2.5 and to the anonymous referee for a key idea.

References

[1] M. Atapour, A. Khodkar, and S. Sheikholeslami, *Characterization of double domination subdivision number of trees,* Discrete Applied Mathematics 155 (2007), 1700–1707.

[2] M. Blidia, M. Chellali, and T. Haynes, *Characterizations of trees with equal paired and double domination numbers,* Discrete Mathematics 306 (2006), 1840–1845.

[3] M. Blidia, O. Favaron, and R. Lounes, *Locating-domination,* 2*-domination and independence in trees,* Australasian Journal of Combinatorics 42 (2008), 309–316.

[4] X. Chen and L. Sun, *Some new results on double domination in graphs,* Journal of Mathematical Research and Exposition 25 (2005), 451–456.

[5] J. Fink, M. Jacobson, *n-domination in graphs,* Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, 282−300.

[6] J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita, and L. Volkmann, *Independence and* 2*-domination in bipartite graphs,* Australasian Journal of Combinatorics 40 (2008), 265–268.

[7] J. Harant and M. Henning, *A realization algorithm for double domination in graphs,* Utilitas Mathematica 76 (2008), 11–24.

[8] F. Harary, T. Haynes, *Double domination in graphs,* Ars Combinatoria 55 (2000), 201–213.

[9] T. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

[10] T. Haynes, S. Hedetniemi, and P. Slater (eds.), *Domination in Graphs: Advanced Topics,* Marcel Dekker, New York, 1998.

[11] Y. Jiao and H. Yu, *On graphs with equal* 2*-domination and connected* 2*-domination numbers,* Mathematica Applicata. Yingyong Shuxue 17 (2004), suppl., 88–92.

[12] R. Shaheen, *Bounds for the* 2*-domination number of toroidal grid graphs,* International Journal of Computer Mathematics 86 (2009), 584–588.

Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80–233 Gdańsk, Poland

E-mail address: `marcin.krzywkowski@gmail.com`