

Bicritical domination and double coalescence of graphs

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Abstract

A graph is bicritical if the removal of any pair of vertices decreases the domination number. We study properties of bicritical graphs and their relation with critical graphs. Next we obtain results for bicritical graphs with edge connectivity two or three. We also generalize the notion of the coalescence of two graphs and we investigate the bicriticality of such graphs.

Keywords: domination, bicritical domination, edge connectivity, coalescence, double coalescence.

AMS Subject Classification: 05C69.

1 Introduction

Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m , thus $|V(G)| = n$ and $|E(G)| = m$. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. Let $\delta(G)$ mean the minimum degree among all vertices of G . A graph is regular if all its vertices have the same degree. A regular graph with vertices of degree k is called a k -regular graph. We say that a graph is cubic if it is 3-regular. The complete graph on n vertices we denote by K_n . The distance between two vertices x and y of a graph G , denoted by $d_G(x, y)$, is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The Cartesian product of

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graphs G and H , denoted by $G \square H$, is a graph such that $V(G \square H) = V(G) \times V(H)$ and $E(G \square H) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1 v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(G)\}$.

The connectivity of a graph G , denoted by $\kappa(G)$, is the minimum size of a subset $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. A graph G is k -connected if its connectivity is at least k . The edge-connectivity of G , denoted by $\lambda(G)$, is the minimum size of a subset $U \subseteq E(G)$ such that $G - U$ is disconnected. A graph is k -edge-connected if its edge-connectivity is at least k .

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D . The domination number of a graph G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . A dominating set of minimum cardinality is called a $\gamma(G)$ -set. For a comprehensive survey of domination in graphs, see [4].

Note that removing a vertex of a graph can increase the domination number by more than one, but can decrease it by at most one. It is useful to consider the set of vertices of a graph as a disjoint union of three sets according to how their removal affects the domination number. We have $V(G) = V^0 \cup V^+ \cup V^-$, where $V^0 = \{v \in V(G) : \gamma(G - v) = \gamma(G)\}$, $V^+ = \{v \in V(G) : \gamma(G - v) > \gamma(G)\}$, $V^- = \{v \in V(G) : \gamma(G - v) < \gamma(G)\}$, for more information see [1–8].

We say that a graph is critical (bicritical, respectively) if removing any vertex (any two vertices, respectively) decreases the domination number. A graph G is k -critical (k -bicritical, respectively) if it is critical (bicritical, respectively) and $\gamma(G) = k$.

We study properties of bicritical graphs and their relation with critical graphs. Next we obtain results for bicritical graphs with edge connectivity two or three. We also generalize the notion of the coalescence of two graphs and we investigate the bicriticality of such graphs.

2 Preliminaries

We begin with the following useful results.

Observation 1 ([1], Observation 5) *Let G be a graph. If $x, y \in V(G)$ are such that $\gamma(G - x - y) = \gamma(G) - 2$, then $d_G(x, y) \geq 3$.*

Proposition 2 ([1], Proposition 2) *For $t \geq 3$, the Cartesian product $G_t = K_t \square K_t$ is t -critical and t -bicritical.*

Proposition 3 ([1], Proposition 14) *If G is a connected bicritical graph, then $\delta(G) \geq 3$.*

Lemma 4 ([1], **Lemma 22**) *Let G be a connected bicritical graph with $\lambda(G) = 2$ and an edge cut $\{ab, cd\}$. Let G_1 and G_2 be the connected components of $G - ab - cd$, with $a, c \in V(G_1)$, $b, d \in V(G_2)$ and $a \neq c$. Then the following must all be true:*

- (i) $\gamma(G) = \gamma(G_1) + \gamma(G_2)$;
- (ii) $a, c \notin V^+(G_1)$ and $b, d \notin V^+(G_2)$;
- (iii) $b \neq d$;
- (iv) without loss of generality, $a, c \in V^-(G_1)$ and $b, d \in V^0(G_2)$;
- (v) neither b nor d is in a $\gamma(G_2)$ -set;
- (vi) $\gamma(G_2 - b - d) = \gamma(G_2) - 1$ and a $\gamma(G_2 - b - d)$ -set dominates neither b nor d ;
- (vii) there is a $\gamma(G_2 - d)$ -set containing b , and there is a $\gamma(G_2 - b)$ -set containing d ;
- (viii) there is no $\gamma(G_1)$ -set containing both a and c ;
- (ix) there is no $\gamma(G_1 - a)$ -set containing c , and there is no $\gamma(G_1 - c)$ -set containing a ;
- (x) $\gamma(G_1) \geq 3$.

Corollary 5 ([1], **Corollary 23**) *Let G be a connected graph. If G is 3-bicritical or 4-bicritical, then $\lambda(G) \geq 3$.*

Theorem 6 ([1], **Theorem 24**) *Let G be a connected bicritical graph. If G is cubic or claw-free, then $\lambda(G) \geq 3$.*

Proposition 7 ([1], **Proposition 25**) *If G is a connected 3-bicritical graph, then $\kappa(G) \geq 3$.*

Proposition 8 ([9], **Proposition 4.1.9**) *If G is a connected graph, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proposition 9 ([9], **Proposition 4.1.11**) *Let G be a connected graph. If G is cubic, then $\kappa(G) = \lambda(G)$.*

3 Results

We now give examples of graphs, which are: both critical and bicritical, only critical, only bicritical, or neither critical nor bicritical.

Let S be a subset of $\mathbb{Z}_n \setminus \{0\}$ such that $x \in S$ implies $-x \in S$. The circulant graph with distance set S is the graph $Circ(n, S)$ with vertex set \mathbb{Z}_n , in which vertices x and y are adjacent if and only if $x - y \in S$.

Proposition 10 *The circulant graph $Circ(9, \{1, 3\})$ is 3-critical and 3-bicritical.*

Proof. Let $G = Circ(9, \{1, 3\})$, see Figure 1. The graph G has the domination number 3 and $\{v_0, v_1, v_2\}$ is a $\gamma(G)$ -set. Since G is vertex-transitive and $\{v_4, v_5\}$ is a dominating set of $G - v_0$, the graph G is critical. Furthermore, $\{v_4, v_5\}$ is a dominating set of the graphs $G - v_0 - v_1$, $G - v_0 - v_2$ and $G - v_0 - v_3$, while $\{v_2, v_6\}$ is a dominating set of the graph $G - v_0 - v_4$. This implies that G is bicritical. ■

The Cartesian product $G_t = K_t \square K_t$ ($t \geq 3$) contains t disjoint copies of K_t in rows and t disjoint copies of K_t in columns, where the vertices of the i th column are $\{v_{i1}, v_{i2}, \dots, v_{it}\}$. Let H be a graph obtained from G_t by adding a new vertex x and joining it to the vertices v_{11}, v_{12}, v_{23} and v_{33} (see Figure 1 (b) for $t = 3$).

Proposition 11 *The graph H obtained from the Cartesian product $G_t = K_t \square K_t$ by adding a new vertex x and joining it to the vertices v_{11}, v_{12}, v_{23} and v_{33} is t -bicritical but is not critical.*

Proof. By Proposition 2, the graph G_t is t -critical. It is easy to observe that $\gamma(G_t) = t = \gamma(H)$ and $\{v_{11}, v_{12}, \dots, v_{1t}\}$ is a γ -set of both those graphs. Thus $x \in V^0(H)$ and the graph H is not critical. We now remove two vertices of H . If we remove x and v_{ij} , then we get $\gamma(H - x - v_{ij}) = t - 1$ as the graph G_t is critical. Now let us remove vertices v_{ij} and v_{kl} . It suffices to consider only the cases when $v_{ij} = v_{11}$ and $v_{kl} \in \{v_{12}, v_{21}, v_{22}, v_{33}\}$. If $v_{kl} = v_{12}$ or $v_{kl} = v_{21}$, then $\{v_{22}, v_{33}, \dots, v_{tt}\}$ is a dominating set of $H - v_{11} - v_{kl}$. If $v_{kl} = v_{22}$, then $\{v_{23}, v_{34}, \dots, v_{(t-1)t}, v_{t2}\}$ is a dominating set of $H - v_{11} - v_{22}$. If $v_{kl} = v_{33}$, then $\{v_{12}, v_{21}, v_{44}, \dots, v_{tt}\}$ is a dominating set of $H - v_{11} - v_{33}$. Thus the graph H is bicritical. ■

Place n vertices around a circle, equally spaced. For even $k \leq n$, the Harary graph $H_{k,n}$ is obtained by making each one of those n vertices adjacent to the nearest $k/2$ vertices in each direction around the circle (see Figure 1 (c), (d) for $k = 4$ and $n = 11, 12$).

Proposition 12 *Let G be a Harary graph $H_{2m,n}$. If $n = (2m + 1)t + 1$, then G is $(t + 1)$ -critical and is not bicritical. If $n = (2m + 1)t + 2$, then G is neither critical nor bicritical.*

Proof. Every vertex of $H_{2m,n}$ has $2m$ neighbors. Therefore $\gamma(H_{2m,n}) \geq \lceil ((2m+1)t+1)/(2m+1) \rceil = \lceil t+1/(2m+1) \rceil = t+1$. The set $\{v_{m+1}, v_{3m+2}, v_{5m+3}, \dots, v_{(2t-1)m+t}, v_0\}$ is a dominating set of $H_{2m,n}$. This implies that $\gamma(H_{2m,n}) = t+1$.

If $n = (2m+1)t+1$, then the t vertices $v_{m+1}, v_{3m+2}, v_{5m+3}, \dots, v_{(2t-1)m+t}$ form a dominating set of $H_{2m,n} - v_0$. Since Harary graphs are vertex-transitive, the graph $H_{2m,n}$ is critical. It is not bicritical as $\gamma(H_{2m,n} - v_0 - v_{m+1}) = \gamma(H_{2m,n})$.

If $n = (2m+1)t+2$, then the graph $H_{2m,n} - v_i$ has $(2m+1)t+1$ vertices. Each one of them has at most $2m$ neighbors. Therefore $\gamma(H_{2m,n}) \geq t+1$. Consequently, $H_{2m,n}$ is not critical. It is not bicritical as $\gamma(H_{2m,n} - v_0 - v_{m+1}) = \gamma(H_{2m,n})$. ■

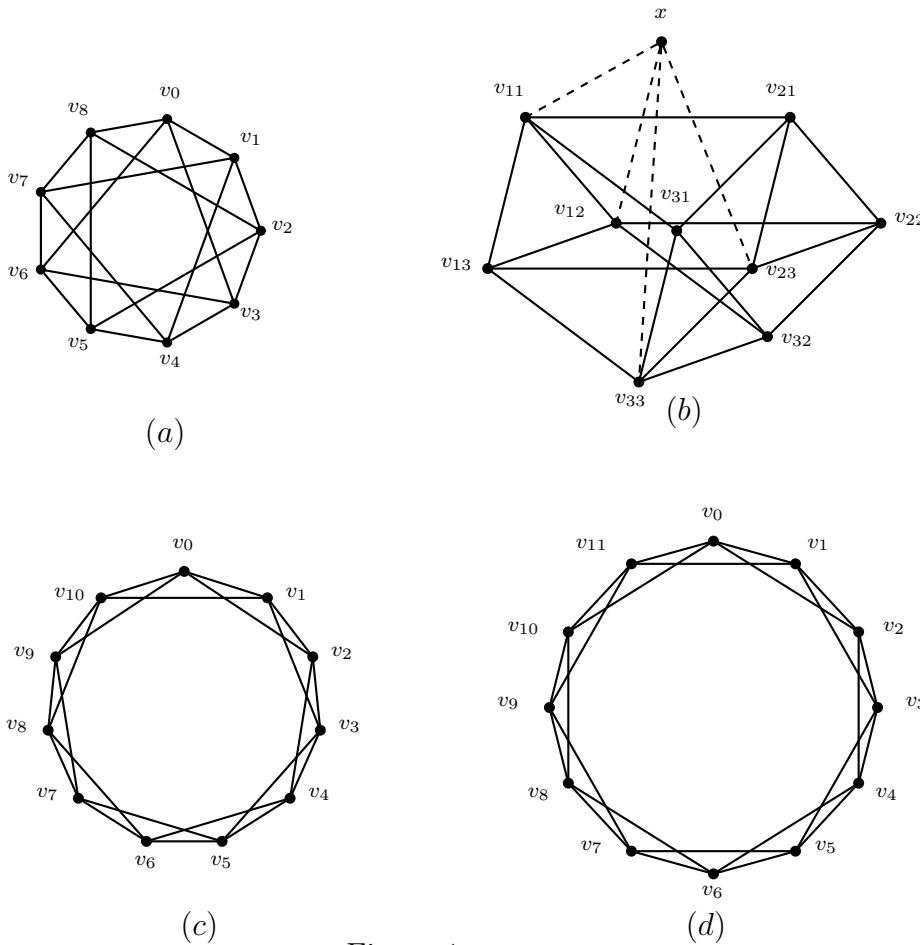


Figure 1

- (a) The circulant graph $Circ(9, \{1, 3\})$
- (b) A graph obtained from $K_3 \square K_3$ by adding a new vertex x and joining it to the vertices v_{11}, v_{12}, v_{23} and v_{33}
- (c) Harary graph $H_{4,11}$
- (d) Harary graph $H_{4,12}$

Figure 1

We now study bicritical graphs with edge connectivity two.

Lemma 13 *Let G be a connected bicritical graph with $\lambda(G) = 2$ and an edge cut $\{ab, cd\}$. Let G_1 and G_2 be the connected components of $G - ab - cd$, with $a, c \in V(G_1)$, $b, d \in V(G_2)$ and $a \neq c$. If $a, c \in V^-(G_1)$, then $d_G(b) \geq 4$, $d_G(d) \geq 4$, and $\gamma(G_2) \geq 2$.*

Proof. Lemma 4 ((v) and (vi)) implies that b and d are not adjacent. By Proposition 3 we have $\delta(G) \geq 3$. Thus the vertex b has some two neighbors, say x and y , in $V(G_2)$. Suppose that $d_G(b) = 3$. Since $d_G(x, y) \leq 2$, Observation 1 implies that $\gamma(G - x - y) = \gamma(G) - 1$. Let D be a $\gamma(G - x - y)$ -set. Clearly, $b \notin D$, and consequently, $a \in D$ as the vertex b has to be dominated. By Lemma 4 (i) we have $\gamma(G) = \gamma(G_1) + \gamma(G_2)$. Let $D_i = D \cap V(G_i)$ for $i = 1, 2$. We have $\gamma(G_1) + \gamma(G_2) = |D_1| + |D_2| + 1$. This implies that $\gamma(G_1) \geq |D_1| + 1$ or $\gamma(G_2) \geq |D_2| + 1$. If $\gamma(G_1) \geq |D_1| + 1$, then c is dominated by d . Let us observe that D_1 is a $\gamma(G_1 - c)$ -set. This is a contradiction to Lemma 4 (ix). Now assume that $\gamma(G_2) \geq |D_2| + 1$. It is easy to see that $D_2 \cup \{b\}$ is a $\gamma(G_2)$ -set, a contradiction to Lemma 4 (v). Therefore $d_G(b) \geq 4$. Similarly we get $d_G(d) \geq 4$.

Now suppose that $\gamma(G_2) = 1$. Thus the graph G_2 has a universal vertex, say x . Since b and d are not adjacent, we have $x \neq b, d$. We get $\gamma(G_1) + \gamma(G_2 - b - d) = \gamma(G - b - d) \leq \gamma(G) - 1 = \gamma(G_1) + \gamma(G_2) - 1 = \gamma(G_1)$. This implies that $\gamma(G_2 - b - d) = 0$, and thus $V(G_2) = \{b, d\}$, a contradiction. ■

Using Corollary 5, Theorem 6 and Propositions 7, 8, 9 we get the following observations.

Observation 14 *Let G be a connected 3-bicritical graph. If $\delta(G) = 3$, then $\kappa(G) = \lambda(G) = 3$.*

Observation 15 *Let G be a connected 4-bicritical graph. If $\delta(G) = 3$, then $\lambda(G) = 3$.*

Theorem 16 *If G is a connected bicritical cubic graph, then $\kappa(G) = \lambda(G) = 3$.*

Proof. By Theorem 6 we have $\lambda(G) \geq 3$. On the other hand, by Proposition 8 we have $\lambda(G) \leq \delta(G) = 3$. Now, using Proposition 9 we get $\kappa(G) = \lambda(G) = 3$. ■

We now investigate bicritical graphs with edge connectivity three.

Proposition 17 *Let G be a connected bicritical graph with $\lambda(G) = 3$ and an edge cut $\{ab, cd, ef\}$. Let G_1 and G_2 be the connected components of $G - ab - cd - ef$, with $a, c, e \in V(G_1)$, $b, d, f \in V(G_2)$ and a, b, c pairwise distinct. Then:*

(i) *If b, d, f are not pairwise distinct, then $\gamma(G) = \gamma(G_1) + \gamma(G_2)$.*

(ii) If $a, c, e \in V^-(G_1)$, then none of the vertices b, d and f is in any $\gamma(G_2)$ -set.

Proof. First assume that the vertices b, d, f are not pairwise distinct. Clearly, $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$. It suffices to show that $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$. Since b, d, f are not pairwise distinct, notice that these symbols denote one or two (distinct) vertices. In the latter case, without loss of generality we can assume that $b = d \neq f$, see Figure 2(a). Using Observation 1 and the fact that G is bicritical, we get $\gamma(G) - 1 = \gamma(G - b - f) = \gamma(G_1) + \gamma(G_2 - b - f) \geq \gamma(G_1) + \gamma(G_2) - 1$, and consequently, $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$. Now assume that $b = d = f$, see Figure 2(b). Let us observe that the vertex b as a neighbor $x \in V(G_2)$, otherwise the graph G is disconnected, a contradiction. We get $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$ similarly as in the previous case (substituting f with x). Therefore $\gamma(G) = \gamma(G_1) + \gamma(G_2)$.

Now assume that $a, c, e \in V^-(G_1)$. Suppose that there exists a $\gamma(G_2)$ -set, say D_2 , which contains some of the vertices b, d, f . Without loss of generality, let it be the vertex b . Let D_1 be a $\gamma(G_1 - a)$ -set. Since $a \in V^-(G_1)$, we have $|D_1| = \gamma(G_1) - 1$. It is easy to observe that $D_1 \cup D_2$ is a dominating set of the graph G . Thus $\gamma(G) \leq |D_1 \cup D_2|$. We now get $\gamma(G) \leq |D_1| + |D_2| = \gamma(G_1) + \gamma(G_2) - 1$, a contradiction. ■

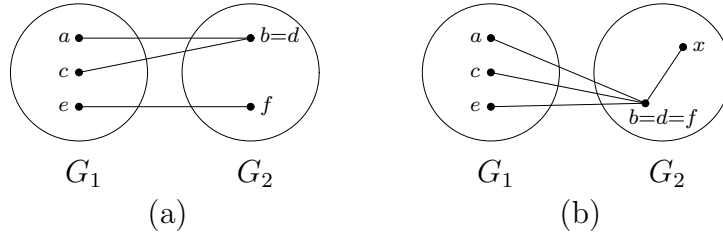


Figure 2

We now generalize the notion of coalescence of graphs.

Definition 18 Let F and H be graphs. Let u_1, v_1 be two adjacent vertices of F and let u_2, v_2 be two adjacent vertices of H . Then $(F \cdot H)(u_1, u_2 : u ; v_1, v_2 : v)$ denotes the graph obtained from F and H by identifying the vertices u_1 and u_2 in a vertex labeled u , and the vertices v_1 and v_2 in a vertex labeled v . We call $(F \cdot H)(u_1, u_2 : u ; v_1, v_2 : v)$ double coalescence of F and H .

We have the following property of the double coalescence of graphs.

Lemma 19 Let G be a double coalescence of graphs F and H . Then

$$\gamma(G) \in \{\gamma(F) + \gamma(H) - 2, \gamma(F) + \gamma(H) - 1, \gamma(F) + \gamma(H)\}.$$

Proof. Let $G = (F \cdot H)(u_1, u_2: u ; v_1, v_2: v)$. Clearly, $\gamma(G) \leq \gamma(F) + \gamma(H)$. Let D be any $\gamma(G)$ -set. If neither u nor v belongs to the set D , then observe that $D \cup \{u_1, u_2\}$ is a dominating set of the graph $F \cup H$. Now assume that exactly one of those vertices, say u , belongs to the set D . Let us observe that $D \setminus \{u\} \cup \{u_1, u_2\}$ is a dominating set of the graph $F \cup H$. If $u, v \in D$, then it is easy to observe that $D \setminus \{u, v\} \cup \{u_1, v_1, u_2, v_2\}$ is a dominating set of the graph $F \cup H$. We now conclude that $\gamma(F \cup H) \leq \gamma(G) + 2$, that is, $\gamma(G) \geq \gamma(F) + \gamma(H) - 2$. ■

Proposition 20 *Let $G = (F \cdot H)(u_1, u_2: u ; v_1, v_2: v)$ be a double coalescence of graphs F and H . Assume that both F and H are bicritical. Then:*

- (i) *If some $\gamma(G)$ -set does not contain any of the vertices u and v , and is a dominating set of $F \cup H$, then the graph G is bicritical.*
- (ii) *If some $\gamma(G)$ -set contains exactly one of the vertices u and v , then the graph G is bicritical.*
- (iii) *If every $\gamma(G)$ -set contains both vertices u and v , then G is not bicritical.*

Proof. Let D be a $\gamma(G)$ -set. First assume that $u, v \notin D$ and D is a dominating set of $F \cup H$. Then we have $\gamma(F) + \gamma(H) = \gamma(F \cup H) \leq |D| = \gamma(G)$. Now assume that the set D contains exactly one of the vertices u and v . Without loss of generality we assume that $u \in D$ and $v \notin D$. Let $D_F = V(F) \cap D \setminus \{u\} \cup \{u_1\}$ and $D_H = V(H) \cap D \setminus \{u\} \cup \{u_2\}$. Let us observe that D_F and D_H are dominating sets of the graphs F and H , respectively. Thus we get $\gamma(F) + \gamma(H) \leq |D_F| + |D_H| = |D| + 1 = \gamma(G) + 1$. Let $x, y \in V(G)$. It suffices to consider the following four possibilities:

- (i) $x, y \in V(F)$ (so $\{x, y\} \cap \{u, v\} = \emptyset$). Then we get $\gamma(G - x - y) \leq \gamma(F - x - y) + \gamma(H - u_2 - v_2) \leq \gamma(F) - 1 + \gamma(H) - 1 = \gamma(F) + \gamma(H) - 2 \leq \gamma(G) - 1$.
- (ii) $x \in V(F), y = u$. Then $\gamma(G - x - y) \leq \gamma(F - x - u_1) + \gamma(H - u_2 - v_2) \leq \gamma(F) - 1 + \gamma(H) - 1 = \gamma(F) + \gamma(H) - 2 \leq \gamma(G) - 1$.
- (iii) $x \in V(F), y \in V(H)$. Then $\gamma(G - x - y) \leq \gamma(F - x - u_1) + \gamma(H - y - v_2) \leq \gamma(F) - 1 + \gamma(H) - 1 = \gamma(F) + \gamma(H) - 2 \leq \gamma(G) - 1$.
- (iv) $x = u$ and $y = v$. Then $\gamma(G - x - y) = \gamma(F - u_1 - v_1) + \gamma(H - u_2 - v_2) \leq \gamma(F) - 1 + \gamma(H) - 1 = \gamma(F) + \gamma(H) - 2 \leq \gamma(G) - 1$.

This implies that the graph G is bicritical.

Now assume that every $\gamma(G)$ -set contains both vertices u and v . Then there are vertices $x \in N_G(u) \setminus \{v\}$ and $y \in N_G(v) \setminus \{u\}$ such that $x \neq y$. Let us consider the graph $G - u - y$. Suppose that G is bicritical. Hence $\gamma(G - u - y) \leq \gamma(G) - 1$. Let us observe that any $\gamma(G - u - y)$ -set together with the vertex v forms a minimum dominating set of the graph G . This is a contradiction as no $\gamma(G)$ -set contains the vertex v . Therefore the graph G is not bicritical. ■

Example Let $G = (Circ(8, \{1, 4\}) \cdot Circ(8, \{1, 4\}))(u_0, v_0 : u ; u_1, v_1 : v)$ be a double coalescence of two copies of the circulant graph $Circ(8, \{1, 4\})$. We have $\gamma(G) = \gamma(G - v_4 - v_5) = 4$, see Figure 3.

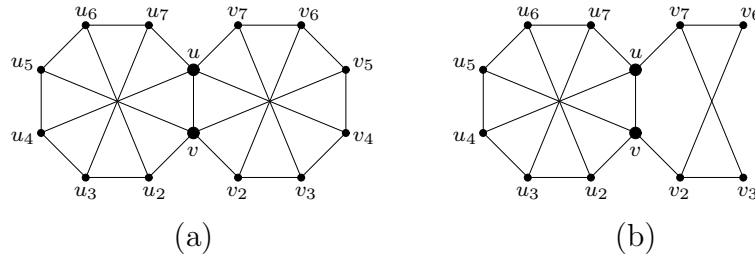


Figure 3

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