An upper bound for the double outer-independent domination number of a tree

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Abstract

A vertex of a graph is said to dominate itself and all of its neighbors. A double outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G is dominated by at least two vertices of D, and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G, denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G. We prove that for every nontrivial tree T of order n, with l leaves and s support vertices we have $\gamma_d^{oi}(T) \leq (2n + l + s)/3$, and we characterize the trees attaining this upper bound.

Keywords: double outer-independent domination, double domination, tree.

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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set.

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A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of G is dominated by at least one vertex of D, while it is a double dominating set of G if every vertex of G is dominated by at least two vertices of D. The domination (double domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_d(G)$, respectively), is the minimum cardinality of a dominating (double dominating, respectively) set of G. Double domination in graphs was introduced by Harary and Haynes [4], and further studied for example in [1, 3]. For a comprehensive survey of domination in graphs, see [5, 6].

A subset $D \subseteq V(G)$ is a double outer-independent dominating set, abbreviated DOIDS, of G if every vertex of G is dominated by at least two vertices of D, and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G, denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G. A double outer-independent dominating set of G of minimum cardinality is called a $\gamma_d^{oi}(G)$-set. The study of double outer-independent domination in graphs was initiated in [7].

A 2-dominating set of a graph G is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D. The 2-domination number of G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. Blidia, Chellali, and Favaron [2] proved the following upper bound on the 2-domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_2(T) \le (n+l)/2$. They also characterized the extremal trees.

We prove the following upper bound on the double outer-independent domination number of a tree. For every nontrivial tree T of order n, with l leaves and s support vertices we have $\gamma_d^{oi}(T) \le (2n+l+s)/3$. We also characterize the trees attaining this upper bound.

2 Results

Since the one-vertex graph does not have a double outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

Observation 1 *Every leaf of a graph G is in every $\gamma_d(G)$-set.*

Observation 2 *Every support vertex of a graph G is in every $\gamma_d(G)$-set.*

We show that if T is a nontrivial tree of order n, with l leaves and s support vertices, then $\gamma_d^{oi}(T)$ is bounded above by $(2n+l+s)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_3 with leaves labeled x and z, and the support vertex labeled y. Let $A(T_1) = \{x, y, z\}$. Let H_1 be a path P_2

with vertices labeled u and v. Let finally H_2 be a path P_3 with leaves labeled u and w, and the support vertex labeled v. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex, say z, by joining it to a support vertex of T_k. Let $A(T_{k+1}) = A(T_k) \cup \{z\}$.
- Operation $\mathcal{O}_2$: Attach a vertex, say z, by joining it to a leaf of T_k adjacent to a strong support vertex. Let $A(T_{k+1}) = A(T_k) \cup \{z\}$.
- Operation $\mathcal{O}_3$: Attach a copy of H_1 by joining the vertex u to a vertex of T_k which is not a leaf and is adjacent to a support vertex. Let $A(T_{k+1})$ $= A(T_k) \cup \{u, v\}$.
- Operation $\mathcal{O}_4$: Attach a copy of H_2 by joining the vertex u to a leaf of T_k adjacent to a weak support vertex. Let $A(T_{k+1}) = A(T_k) \cup \{v, w\}$.

We now prove that for every tree T of the family $\mathcal{T}$, the set $A(T)$ defined above is a DOIDS of minimum cardinality equal to $(2n + l + s)/3$.

Lemma 3 *If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_d^{oi}(T)$-set of size $(2n + l + s)/3$.*

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graphs H_1 and H_2 defined above. To show that $A(T)$ is a $\gamma_d^{oi}(T)$-set of cardinality $(2n+l+s)/3$ we use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_3$, then $(2n+l+s)/3$ $= (6 + 2 + 1)/3 = 3 = |A(T)| = \gamma_d^{oi}(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. For a given tree T', let n' denote its order, l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n'+1$, $l = l' + 1$ and $s = s'$. The vertex to which is attached z we denote by x. Let y be a leaf adjacent to x and different from z. By Observation 2 we have $x \in A(T')$. It is easy to see that $A(T) = A(T') \cup \{z\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$. Now let D be any $\gamma_d^{oi}(T)$-set. By Observations 1 and 2 we have $z, y, x \in D$. It is easy to see that $D \setminus \{z\}$ is a DOIDS of the tree T'. Therefore $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 1$. We now conclude that $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1$. We get $\gamma_d^{oi}(T) = |A(T)| = |A(T')|+1 = (2n'+l'+s')/3+1 = (2n-2+l-1+s)/3+1$ $= (2n + l + s)/3$.

Now suppose that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n'+1$, $l = l'$ and $s = s' + 1$. The leaf to which is attached z we denote by x. By y we denote the neighbor of x other than z. By Observation 1 we have $x \in A(T')$.

It is easy to see that $A(T) = A(T') \cup \{z\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$. Now let D be any $\gamma_d^{oi}(T)$-set. By Observations 1 and 2 we have $z, x, y \in D$. It is easy to see that $D \setminus \{z\}$ is a DOIDS of the tree T'. Therefore $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 1$. We now conclude that $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 1$. We get $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 1 = (2n' + l' + s')/3 + 1 = (2n - 2 + l + s - 1)/3 + 1$ $= (2n + l + s)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n' + 2$, $l = l' + 1$ and $s = s' + 1$. The vertex to which is attached P_2 we denote by x. Let y be a support vertex adjacent to x, and let z be a leaf adjacent to y. Obviously, $A(T) = A(T') \cup \{u, v\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$. Now let D be any $\gamma_d^{oi}(T)$-set. By Observations 1 and 2 we have $v, z, u, y \in D$. If $x \in D$, then it is easy to see that $D \setminus \{u, v\}$ is a DOIDS of the tree T'. Now suppose that $x \notin D$. Let a denote a neighbor of x other than u and y. The set $V(T) \setminus D$ is independent, thus $a \in D$. Let us observe that now also $D \setminus \{u, v\}$ is a DOIDS of the tree T' as the vertex x is still dominated at least twice. Therefore $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$. We now conclude that $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$. We get $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 2 = (2n' + l' + s')/3 + 2 = (2n - 4 + l - 1 + s - 1)/3 + 2$ $= (2n + l + s)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_4$. We have $n = n' + 3$, $l = l'$ and $s = s'$. The leaf to which is attached P_3 we denote by x. By Observation 1 we have $x \in A(T')$. It is easy to see that $D' \cup \{v, w\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d^{oi}(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 1 and 2 we have $w, v \in D$. Observe that $D \setminus \{v, w\}$ is a DOIDS of the tree T'. Therefore $\gamma_d^{oi}(T') \leq \gamma_d^{oi}(T) - 2$. We now conclude that $\gamma_d^{oi}(T) = \gamma_d^{oi}(T') + 2$. We get $\gamma_d^{oi}(T) = |A(T)| = |A(T')| + 2 = (2n' + l' + s')/3 + 2 = (2n - 6 + l + s)/3 + 2$ $= (2n + l + s)/3$. ■

We now establish the main result, an upper bound on the double outer-independent domination number of a tree together with the characterization of the extremal trees.

Theorem 4 *If T is a tree of order n, with l leaves and s support vertices, then $\gamma_d^{oi}(T) \leq (2n + l + s)/3$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $\gamma_d^{oi}(T) = 2 < (4 + 2 + 2)/3$ $= (2n + l + s)/3$. Now suppose that $\mathrm{diam}(T) \geq 2$. Thus the order n of the tree T is at least three. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$, with l' leaves and s' support vertices.

First suppose that some support vertex of T, say x, is strong. Let y and z be leaves adjacent to x. Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observation 2 we have $x \in D'$. It is easy to see that $D' \cup \{y\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 1$. We now get $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T')+1 \le (2n'+l'+s')/3+1 = (2n-2+l-1+s)/3+1 = (2n+l+s)/3$. If $\gamma_d^{oi}(T) = (2n + l + s)/3$, then obviously $\gamma_d^{oi}(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity diam(T). Let t be a leaf at maximum distance from r, and let v be the parent of t in the rooted tree. If diam$(T) \ge 3$, then let u be the parent of v. If diam$(T) \ge 4$, then let w be the parent of u. If diam$(T) \ge 5$, then let d be the parent of w. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First suppose that $d_T(u) \ge 3$. Assume that among the children of u there is a support vertex, say x, different from v. The leaf adjacent to x we denote by y. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Let D' be any $\gamma_d^{oi}(T')$-set. Obviously, $D' \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 2$. We now get $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 2 \le (2n' + l' + s')/3 + 2$ $= (2n - 4 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3$. If $\gamma_d^{oi}(T) = (2n + l + s)/3$, then $\gamma_d^{oi}(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that some child of u, say x, is a leaf. Let $T' = T - t$. We have $n' = n - 1$, $l' = l$ and $s' = s - 1$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observation 1 we have $v \in D'$. It is easy to see that $D' \cup \{t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 1$. We now get $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 1 \le (2n' + l' + s')/3 + 1$ $= (2n - 2 + l + s - 1)/3 + 1 = (2n + l + s)/3$. If $\gamma_d^{oi}(T) = (2n + l + s)/3$, then $\gamma_d^{oi}(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

If $d_T(u) = 1$, then $T = P_3 = T_1 \in \mathcal{T}$. By Lemma 3 we have $\gamma_d^{oi}(T) = (2n$ $+l + s)/3$. Now consider the case when $d_T(u) = 2$. First assume that there is a child of w other than u, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_d^{oi}(T')$-set that does not contain the vertex k. Let D' be such a set. The set $V(T') \setminus D'$ is independent, thus $w \in D'$. It is easy to observe that $D' \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 2$. We now get $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 2 \le (2n' + l' + s')/3 + 2 = (2n - 6 + l - 1 + s - 1)/3 + 2$ $= (2n + l + s)/3 - 2/3 < (2n + l + s)/3$.

Now suppose that w is adjacent to a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observation 2 we have $w \in D'$. It is easy to observe that $D' \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 2$. We now get $\gamma_d^{oi}(T) \le \gamma_d^{oi}(T') + 2 \le (2n' + l' + s')/3 + 2$ $= (2n - 6 + l - 1 + s - 1)/3 + 2 = (2n + l + s)/3 - 2/3 < (2n + l + s)/3$. Henceforth, we can assume that w is not adjacent to any leaf.

Now suppose that there is a child of w, say k, such that the distance of w to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T' = T - T_u - l$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observations 1 and 2 we have $k, w \in D'$. It is easy to observe that $D' \cup \{v, t, l\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 3$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 3 \leq (2n' + l' + s')/3 + 3 = (2n - 8 + l - 1 + s - 1)/3 + 3 = (2n + l + s)/3 - 1/3 < (2n + l + s)/3$.

If $d_T(w) = 1$, then $T = P_4$. We have $T \in \mathcal{T}$ as it can be obtained from P_3 by operation $\mathcal{O}_2$. By Lemma 3 we have $\gamma_d^{oi}(T) = (2n + l + s)/3$. Now consider the case when $d_T(w) = 2$. Let $T' = T - T_u$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observation 1 we have $w \in D'$. It is easy to see that $D' \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$. First suppose that d is adjacent to a leaf. We have $n' = n - 3$, $l' = l$ and $s' = s - 1$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l + s - 1)/3 + 2 = (2n + l + s)/3 - 1/3 < (2n + l + s)/3$.

Now assume that no neighbor of d is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l$ and $s' = s$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq (2n' + l' + s')/3 + 2 = (2n - 6 + l + s)/3 + 2 = (2n + l + s)/3$. If $\gamma_d^{oi}(T) = (2n + l + s)/3$, then $\gamma_d^{oi}(T') = (2n' + l' + s')/3$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$. ■

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