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Hat Problem on the Cycle C_4

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Abstract

The topic of our paper is the hat problem. In that problem, each of n people is randomly fitted with a blue or red hat. Then everybody can try to guess simultanously his own hat color looking at the hat colors of the other people. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win. In this version every person can see everybody excluding him. We consider such problem on a graph, where vertices are people, and a person can see these people to which he is connected by an edge. The solution of the hat problem is known for trees. In this paper we solve the problem on the cycle C_4.

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1 Introduction

In the hat problem, a team of n people enters a room and a blue or red hat is randomly placed on the head of each person. Each person can see the hats of all of the other people but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each person must simultanously guess the color of his own hat or pass. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win.

The hat problem with seven people called "seven prisoners puzzle" was formulated by T. Ebert in his Ph.D. Thesis [1]. The hat problem with three people was the subject of an article in The New York Times [3].

The hat problem has many applications and connections to other areas of science, for example: information technology, linear programming, genetic programming, economy, biology, approximating Boolean functions, and autoreducibility of random sequences. Therefore, it is hoped that the hat problem on a graph considered in this paper, as a natural generalization, is worth exploring, and may also have many applications.

In the hat problem on a graph, vertices are people and a person can see these people, to which he is connected by an edge. This variant of the hat problem was first considered in [2] where there are proved some general theorems about the hat problem on a graph, and the problem is solved on trees.

In this paper we solve the hat problem on the cycle with four vertices.

2 Preliminaries

For a graph G, by $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges of this graph, respectively. If H is a subgraph of G, then we write $H \subseteq G$. Let $v \in V(G)$. By $N_G(v)$ we denote the neighbourhood of v, that is $N_G(v) = \{x \in V(G)\colon vx \in E(G)\}$. By P_n (C_n, K_n, respectively) we denote the path (cycle, complete graph, respectively) with n vertices.

Without loss of generality we may assume an ordering of the vertices of a graph G, that is $V(G) = \{v_1, v_2, \ldots, v_n\}$.

If $v_i \in V(G)$, then $c(v_i)$ is the first letter of the color of v_i, so $c\colon V(G) \to \{b, r\}$ is a function. By a case for the graph G we mean a sequence $(c(v_1), c(v_2), \ldots, c(v_n))$. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

If $v_i \in V(G)$, then by s_i we denote a function $s_i\colon V(G) \to \{b, r, *\}$, where $s_i(v_j)$ is the first letter of the color of v_j if v_i sees v_j, and mark $*$ otherwise, that is, $s_i(v_j) = c(v_j)$ if $v_j \in N_G(v_i)$, while $s_i(v_j) = *$ if $v_j \in V(G) \setminus N_G(v_i)$. By a situation of the vertex v_i in the graph G we mean the sequence $(s_i(v_1), s_i(v_2), \ldots, s_i(v_n))$. The set of all possible situations of v_i in the graph G we denote $St_i(G)$. Of course, $|St_i(G)| = 2^{|N_G(v_i)|}$.

Let $v_i \in V(G)$. We say that a case $(c_1, c_2, \ldots, c_n)$ for the graph G corresponds to a situation $(t_1, t_2, \ldots, t_n)$ of the vertex v_i in the graph G if it is created from this situation only by changing every mark $*$ to the letter b or r. So, a case corresponds to a situation of v_i if every vertex adjacent to v_i, in that case case has the same color as in that situation. To every situation of the vertex v_i in the graph G correspond $2^{|V(G)|-d_G(v_i)}$ cases, because every situation of v_i has $|V(G)| - d_G(v_i)$ marks $*$.

By a statement of a vertex we mean its declaration about the color it guesses it is. By the effect of a case we mean a win or a loss. According to the definition of the hat problem, the effect of a case is a win if at least one vertex states its color correctly and no vertex states its color wrong. The effect of a case is a loss if no vertex states its color or somebody states its color wrong.

By a guessing instruction for the vertex $v_i \in V(G)$ (denoted by g_i) we mean a function $g_i\colon St_i(G) \to \{b, r, p\}$ which, for a given situation, gives the first letter of the color v_i guesses it is or a letter p if v_i passes. Thus a guessing instruction is a rule which determines the conduct of the vertex v_i in every situation. By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$. By $\mathcal{F}(G)$ we denote the family of all strategies for the graph G.

Let $v_i \in V(G)$ and $S \in \mathcal{F}(G)$. We say that v_i never states its color in the strategy S if v_i passes in every situation, that is $g_i \equiv p$. We say that v_i always states its color in the strategy S if v_i states its color in every situation, that is, for every $T \in St_i(G)$ we have $g_i(T) \in \{b, r\}$ ($g_i(T) \neq p$, equivalently).

If $S \in \mathcal{F}(G)$, then by $Cw(S)$ and $Cl(S)$ we denote the sets of cases for the graph G in which the team wins or loses, respectively. Of course, $|Cw(S)| + |Cl(S)| = |C(G)|$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = \frac{|Cw(S)|}{|C(G)|}$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. Certainly $p(S) \leq h(G)$. We say that the strategy S is optimal for the graph G if $p(S) = h(G)$. By $\mathcal{F}^0(G)$ we denote the family of all optimal strategies for the graph G.

Let $t, m_1, m_2, \ldots, m_t \in \{1, 2, \ldots, n\}$ be such that $m_j \neq m_k$ and $c_{m_j} \in \{b, r\}$, for every $j, k \in \{1, 2, \ldots, t\}$. By $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ we denote the set of cases for the graph G such that the first letter of the color of v_{m_j} is c_{m_j}.

The following theorems are from [2]. The first of them presents a relation between the hat number of a graph and the hat number of its any subgraph.

Theorem 1 *If H is a subgraph of G, then $h(H) \leq h(G)$.*

Since the graph K_1 is a subgraph of every graph, we get the following Corollary.

Corollary 2 *For every graph G we have $h(G) \geq \frac{1}{2}$.*

In the next two theorems there are considered optimal strategies such that some vertex always (never, respectively) states its color.

Theorem 3 *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v always states its color, then $h(G) = \frac{1}{2}$.*

Theorem 4 *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v never states its color, then $h(G) = h(G - v)$.*

The following theorem is the solution of the hat problem on paths.

Theorem 5 *For every path P_n we have $h(P_n) = \frac{1}{2}$.*

The next fact is about the unnecessity of statements of any further vertices in a case in which some vertex already states its color.

Fact 6 *Let G be a graph and let S be a strategy for this graph. Let C be a case in which some vertex states its color. Then a statement of any other vertex cannot improve the effect of the case C.*

Now we characterize the number of cases in which the loss of the team is caused by a statement of a vertex.

Fact 7 *Let G be a graph and let v_i be a vertex of G. Let $S \in \mathcal{F}(G)$. If v_i states its color in a situation, then the team loses in at least half of all cases corresponding to this situation.*

3 Results

In the following theorem we solve the hat problem on the cycle with four vertices.

Theorem 8 $h(C_4) = \frac{1}{2}$.

Proof. Let S be an optimal strategy for C_4 such that there is no situation in which both v_1 and v_3 state its colors, and there is no situation in which both v_2 and v_4 state its colors. Now we prove that such strategy exists. Let S' be an optimal strategy for C_4. Assume in S' there is a situation in which both v_1 and v_3 state its colors, or there is a situation in which both v_2 and v_4 state its colors. Let the strategy S differ from S' only by that v_3 does not state its color when v_1 states its color, and v_4 does not state its color when v_2 states its color. By Fact 6 the statements of v_3 and v_4 cannot improve the effect of any from that cases. Therefore, $p(S) \geq p(S')$. Since $S' \in \mathcal{F}^0(C_4)$, the strategy S is also optimal. In the strategy S there is no such situation in which both v_1 and v_3 state its colors, and there is no such situation in which both v_2 and v_4 state its colors. If some vertex in C_4 never states its color, then let $i \in \{1, 2, 3, 4\}$ be such that v_i never states its color. By Theorem 4 we have $h(C_4) = h(C_4 - v_i)$. Since $C_4 - v_i = P_3$, and by Theorem 5 we have $h(P_3) = \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$. Now assume every vertex in C_4 states its color. If some vertex in C_4 always states its color, then by Theorem 3 we have $h(C_4) = \frac{1}{2}$. Now assume there is no vertex in C_4 such that always states its color. Every vertex states its color in one, two, or three situations. We consider the following two possibilities: (1) every vertex states its color in exactly one situation; (2) there is a vertex which states its color in at least two situations.

(1) Any statement of any vertex in any situation is correct in exactly two cases, because to every situation of any vertex correspond four cases, and in

the half of them this vertex has the color it states it has. If every vertex states its color in exactly one situation, then there are exactly 8 correct statements, and even if every of them is in another case, then the team can win in at most 8 cases. This implies that $p(S) \le \frac{8}{16} = \frac{1}{2}$. Since $S \in \mathcal{F}^0(C_4)$, we have $h(C_4) \le \frac{1}{2}$. Since by Corollary 2 we have $h(C_4) \ge \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$.

(2) We consider the following two possibilities: (2.1) there is a vertex which states its color in exactly three situations; (2.2) every vertex states its color in at most two situations.

(2.1) Without loss of generality we assume v_1 states its color in exactly three situations. Since there is no such situation in which both v_1 and v_3 state its colors, and v_3 states its color in at least one situation, v_3 states its color in exactly one situation. Since in every from the situations $(*, b, *, b)$, $(*, b, *, r)$, $(*, r, *, b)$, and $(*, r, *, r)$ the vertex v_1 or v_3 states his hat color, by Fact 7 we have

$$|Cl(S, v_2^b, v_2^b)| \ge \frac{|C(C_4, v_2^b, v_2^b)|}{2}, \quad |Cl(S, v_2^b, v_2^r)| \ge \frac{|C(C_4, v_2^b, v_2^r)|}{2},$$

$$|Cl(S, v_2^r, v_2^b)| \ge \frac{|C(C_4, v_2^r, v_2^b)|}{2}, \quad \text{and} \quad |Cl(S, v_2^r, v_2^r)| \ge \frac{|C(C_4, v_2^r, v_2^r)|}{2}.$$

Consequently

$$|Cl(S)| = |Cl(S, v_2^b, v_2^b)| + |Cl(S, v_2^b, v_2^r)| + |Cl(S, v_2^r, v_2^b)| + |Cl(S, v_2^r, v_2^r)|$$

$$\ge \frac{|C(C_4, v_2^b, v_2^b)|}{2} + \frac{|C(C_4, v_2^b, v_2^r)|}{2} + \frac{|C(C_4, v_2^r, v_2^b)|}{2} + \frac{|C(C_4, v_2^r, v_2^r)|}{2} = \frac{|C(C_4)|}{2}.$$

Now we get

$$p(S) = \frac{|Cw(S)|}{|C(C_4)|} = \frac{|C(C_4)| - |Cl(S)|}{|C(C_4)|} \le \frac{|C(C_4)| - \frac{|C(C_4)|}{2}}{|C(C_4)|} = \frac{1}{2}.$$

Since $S \in \mathcal{F}^0(C_4)$, we have $h(C_4) \le \frac{1}{2}$. Since by Corollary 2 we have $h(C_4) \ge \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$.

(2.2) Since there is a vertex which states its color in exactly two situations, without loss of generality we assume v_1 states its color in exactly two situations. We consider the following two possibilities: (2.2.1) v_3 states its color in exactly two situations; (2.2.2) v_3 states its color in exactly one situation.

(2.2.1) Since in every from the situations $(*, b, *, b)$, $(*, b, *, r)$, $(*, r, *, b)$, $(*, b, *, b)$ v_1 or v_3 states his hat color, by the same arguments as in (2.1), we get $h(C_4) = \frac{1}{2}$.

(2.2.2) We consider the following two possibilities: (a1) in both situations in which v_1 states its color, v_2 has the same color or v_4 has the same color; (a2) in both situations in which v_1 states its color, v_2 has different colors, and v_4

has different colors. We consider the following two possibilities: (b1) in both situations v_1 states it has the same color; (b2) in both situations v_1 states it has different colors.

Let $v_i \in \{v_1, v_2\}$. Now we consider the following four possibilities: (2.2.2.1) (a1),(b1); (2.2.2.2) (a1),(b2); (2.2.2.3) (a2),(b1); (2.2.2.4) (a2),(b2).

(2.2.2.1) Without loss of generality we assume v_1 states its color in the situations $(*, b, *, b)$ and $(*, b, *, r)$, and in these situations it states it is blue. Also without loss of generality we assume v_3 states its color in the situation $(*, r, *, b)$, and in this situation it states it is blue. These statements are correct in the cases: (b, b, b, b), (b, b, r, b), (b, b, b, r), (b, b, r, r), (b, r, b, b), and (r, r, b, b), and are wrong in the cases: (r, b, b, b), (r, b, r, b), (r, b, b, r), (r, b, r, r), (b, r, r, b), and (r, r, r, b),. To the situation $(b, *, b, *)$ correspond three cases in which v_1 or v_3 states its color correctly, and the case (b, r, b, r) in which neither v_1 nor v_3 states its color. By Fact 6, among cases corresponding to the situation $(b, *, b, *)$, the effect only of (b, r, b, r) can be improved. In two cases corresponding to the situation $(b, *, b, *)$ the statement of v_i is wrong. This implies that in at least one case corresponding to the situation $(b, *, b, *)$ in which v_1 or v_3 states its color correctly, v_i states its color wrong. Therefore, the statement of v_i in the situation $(b, *, b, *)$ cannot improve the chance of success. Thus we assume v_i does not states its color in the situation $(b, *, b, *)$. Now let us consider the cases corresponding to the situation $(b, *, r, *)$. To the situation $(b, *, r, *)$ correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which v_i does not state its color. By Fact 6, among cases corresponding to the situation $(b, *, r, *)$, the effect only of (b, r, r, r) can be improved. To improve the effect of this case, the statement of v_i has to be correct in this case. Among four cases corresponding to the situation $(b, *, r, *)$ in two of them the statement of v_i is wrong. This implies that in some case corresponding to the situation $(b, *, b, *)$ in which v_1 or v_3 states its color correctly, v_i states its color wrong, making worse the evaluation of this case. Therefore, the statement of v_i in the situation $(b, *, r, *)$ cannot improve the chance of success. Thus we may assume v_i does not state its color in the situation $(b, *, r, *)$. There is only one case corresponding to the situation $(r, *, b, *)$ in which neither v_1 nor v_3 states its color. There in also only one case corresponding to the situation $(r, *, r, *)$ in which neither v_1 nor v_3 states its color. Therefore, there are only two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S) = \frac{|Cw(S)|}{|C(C_n)|} \leq \frac{8}{16} = \frac{1}{2}$. Since S is an optimal strategy for G, we have $h(C_4) \leq \frac{1}{2}$. Since by Theorem 1 we have $h(C_4) \geq \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$.

(2.2.2.2) Without loss of generality we assume in the situation $(*, b, *, b)$ v_1 states it is blue, and in the situation $(*, b, *, r)$ it states it is red. Also without loss of generality we assume v_3 states its color in the situation $(*, r, *, b)$, and in this situation it states it is blue. These statements are correct in the cases: (b,b,b,b), (b,b,r,b), (r,b,b,r), (r,b,r,r), (b,r,b,b), and (r,r,b,b), and are wrong in the cases: (r,b,b,b), (r,b,r,b), (b,b,b,r), (b,b,r,r), (b,r,r,b), and (r,r,r,b). To the situation $(b,*,b,*)$ correspond two correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 nor v_3 states its color. To the situation $(r,*,b,*)$ also correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 nor v_3 states its color. By reasons similar as when considering the situation $(b,*,r,*)$ in (2.2.2.1), we may assume v_i does not states its color in any of the situations $(b,*,b,*)$ and $(r,*,b,*)$. To the situation $(b,*,r,*)$ correspond three cases in which v_1 or v_3 states its color, and one in which neither v_1 nor v_3 states its color. To the situation $(r,*,r,*)$ also correspond three cases in which v_1 or v_3 states its color, and one in which neither v_1 nor v_3 states its color. Therefore, by Fact 6, there are two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S) = \frac{|Cw(S)|}{|C(C_4)|} \le \frac{8}{16} = \frac{1}{2}$.

(2.2.2.3) Without loss of generality we assume v_1 states its color in the situations $(*,b,*,b)$ and $(*,r,*,r)$, and in these situations it states it is blue. Also without loss of generality we assume v_3 states its color in the situation $(*,r,*,b)$, and in this situation it states it is blue. These statements are correct in the cases: (b,b,b,b), (b,b,r,b), (b,r,b,r), (b,r,r,r), (b,r,b,b), and (r,r,b,b), and are wrong in the cases: (r,b,b,b), (r,b,r,b), (r,r,b,r), (r,r,r,r), (b,r,r,b), and (r,r,r,b). To the situation $(b,*,b,*)$ correspond three cases in which v_1 or v_3 states its color correctly, and one in which neither v_1 nor v_3 states its color. By reasons similar as in the situation $(b,*,b,*)$ in (2.2.2.1), we may assume v_i does not state its color in the situation $(b,*,b,*)$. To the situation $(b,*,r,*)$ correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 nor v_3 states its color. By reasons similar in the situation $(b,*,r,*)$ in (2.2.2.1), we may assume v_i does not states its color in the situation $(b,*,r,*)$. To the situation $(r,*,b,*)$ corresponds only one case in which neither v_1 nor v_3 states its color, and therefore statements of v_2 or v_4 can improve the effects of only two cases. To the situation $(r,*,r,*)$ also corresponds only one case in which neither v_1 nor v_3 states its color. Therefore, by Fact 6, there are two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S) = \frac{|Cw(S)|}{|C(C_4)|} \le \frac{8}{16} = \frac{1}{2}$.

(2.2.2.4) Without loss of generality we assume in the situation $(*,b,*,b)$ v_1 states it is blue, and in the situation $(*,r,*,r)$ it states it is red. Also without loss of generality we assume v_3 states its color in the situation $(*,r,*,b)$, and in this situation it states it is blue. These statements are correct in the cases: (b,b,b,b), (b,b,r,b), (r,r,b,r), (r,r,r,r), (b,r,b,b), and (r,r,b,b), and are wrong in the cases: (r,b,b,b), (r,b,r,b), (b,r,b,r), (b,r,r,r), (b,r,r,b), and (r,r,r,b). To the situation $(b,*,b,*)$ correspond two cases in which v_1 or v_3

states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 or v_3 states its color. To the situation $(r, *, b, *)$ correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 or v_3 states its color. By reasons similar as when considering the situation $(b, *, r, *)$ in (2.2.2.1), we may assume v_i does not states its color in any of the situations $(b, *, b, *)$ and $(r, *, b, *)$. To the situation $(b, *, r, *)$ corresponds only one case in which neither v_1 nor v_3 states its color, and therefore the statements of v_2 or v_4 can improve the effects of only two cases. To the situation $(r, *, r, *)$ corresponds only one case in which neither v_1 nor v_3 states its color. Therefore, by Fact 6, there are two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S) = \frac{|Cw(S)|}{|C(C_4)|} \leq \frac{8}{16} = \frac{1}{2}$, and consequenty $h(C_4) = \frac{1}{2}$. ■

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