An upper bound on the 2-outer-independent domination number of a tree

Borne supérieure sur le nombre de 2-domination extérieurement-indépendante d'un arbre

Marcin Krzywkowski
e-mail: marcin.krzywkowski@gmail.com

Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology
Narutowicza 11/12, 80–233 Gdańsk, Poland

Abstract

A 2-outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has a at least two neighbors in D, and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of a graph G, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G. We prove that for every nontrivial tree T of order n with l leaves we have $\gamma_2^{oi}(T) \leq (n+l)/2$, and we characterize the trees attaining this upper bound.

Résumé

Un ensemble 2-dominant extérieurement-indépendant d'un graphe G est un ensemble D de sommets de G tel que chaque sommet de $V(G) \setminus D$ a au moins deux voisins dans D, et l'ensemble $V(G) \setminus D$ est indépendant. Le nombre de 2-domination extérieurement-indépendante d'un graphe G, noté par $\gamma_t^{oi}(G)$, est le cardinal minimum d'un ensemble 2-dominant extérieurement-indépendant de G. Nous prouvons l'inégalité $\gamma_2^{oi}(T) \leq (n+l)/2$ pour chaque arbre non trivial T d'ordre n avec l feuilles, et nous caractérisons les arbres atteignant cette borne supérieure.

1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). By G^* we denote the graph obtained from G by removing all leaves. The path on n vertices we denote by P_n.

We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of $V(G)$. An independent subset of the set of vertices of G of maximum cardinality is called an $\alpha(G)$-set.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination

(2-domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [3], and further studied for example in [1, 2, 4, 5, 8, 10]. For a comprehensive survey of domination in graphs, see [6, 7].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D, and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G. A 2-outer-independent dominating set of G of minimum cardinality is called a $\gamma_2^{oi}(G)$-set. The study of 2-outer-independent domination in graphs was initiated in [9].

Blidia, Chellali, and Favaron [1] established the following upper bound on the 2-domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_2(T) \leq (n+l)/2$. They also characterized the extremal trees.

We prove the following upper bound on the 2-outer-independent domination number of a tree. For every nontrivial tree T of order n with l leaves we have $\gamma_2^{oi}(T) \leq (n+l)/2$. We also characterize the trees attaining this upper bound.

2 Results

We begin with the following straightforward observation.

Observation 1 *Every leaf of a graph G is in every $\gamma_2^{oi}(G)$-set.*

We have the following relation between the 2-outer-independent domination number of a graph without isolated vertices and the independence number of the graph obtained from it by removing all leaves.

Lemma 2 *If G is a graph without isolated vertices, then $\gamma_2^{oi}(G) = n - \alpha(G^*)$.*

Proof. Let D be any $\gamma_2^{oi}(G)$-set. By Observation 1, all leaves belong to the set D. Therefore $V(G) \setminus D \subseteq V(G^*)$. The set $V(G) \setminus D$ is independent, thus $\alpha(G^*) \geq |V(G) \setminus D| = n - \gamma_2^{oi}(G)$. Now let D^* be any $\alpha(G^*)$-set. Let us observe that in the graph G every vertex of D^* has at least two neighbors in the set $V(G) \setminus D^*$. Therefore $\gamma_2^{oi}(G) \leq |V(G) \setminus D^*| = n - \alpha(G^*)$. This implies that $\gamma_2^{oi}(G) = n - \alpha(G^*)$. ■

Now we get an upper bound on the 2-outer-independent domination number of bipartite graphs without isolated vertices.

Lemma 3 *For every bipartite graph G without isolated vertices of order n with l leaves we have $\gamma_2^{oi}(G) \leq (n+l)/2$.*

Proof. Observe that the graph G^* is also bipartite. Thus there is an independent subset of the set of its vertices which contains at least half of them. Therefore $\alpha(G^*) \geq |V(G^*)|/2 = (n-l)/2$. Using Lemma 2 we get $\gamma_2^{oi}(G) = n - \alpha(G^*) \leq n - (n-l)/2 = (n+l)/2$. ■

By $\mathcal{T}_{max}$ we denote the family of trees whose 2-outer-independent domination number attains the upper bound from the previous lemma.

We have the following property of trees of the family $\mathcal{T}_{max}$.

Lemma 4 *Let T be a tree. We have $T \in \mathcal{T}_{max}$ if and only if $\alpha(T^*) = n^*/2$.*

Proof. If T is a tree of the family $\mathcal{T}_{max}$, that is $\gamma_2^{oi}(T) = (n+l)/2$, then using Lemma 2 we get $\alpha(T^*) = n - \gamma_2^{oi}(T) = n - (n+l)/2 = (n-l)/2 = n^*/2$. The converse implication can be proven similarly. ■

We showed that if G is a bipartite graph without isolated vertices of order n with l leaves, then $\gamma_2^{oi}(G)$ is bounded above by $(n+l)/2$. We characterize all trees attaining this bound. For this purpose we introduce a family $\mathcal{T}$ of trees that can be obtained from P_2 by applying consecutively operations $\mathcal{O}_1$ or $\mathcal{O}_2$ defined below.

- Operation $\mathcal{O}_1$: Add one new vertex and one edge joining this new vertex to a non-leaf vertex of a graph.
- Operation $\mathcal{O}_2$: Add two new vertices, one edge joining them, and one edge joining one of them to a leaf of a graph.

Now we prove that for every tree of the family $\mathcal{T}$, the 2-outer-independent domination number equals the number of leaves plus half of the remaining vertices.

Lemma 5 *Any tree $T \in \mathcal{T}$ is in $\mathcal{T}_{max}$.*

Proof. We have $\gamma_2^{oi}(P_2) = 2 = (2+2)/2 = (n+l)/2$, thus $P_2 \in \mathcal{T}_{max}$. Therefore the result is true for the starting tree. It remains to show that performing the operations $\mathcal{O}_1$ and $\mathcal{O}_2$ keeps the property of being in $\mathcal{T}_{max}$. Let T be a tree obtained from $T' \in \mathcal{T}$ by operation $\mathcal{O}_1$. We have $T^* = T'^*$. If $T' \in \mathcal{T}_{max}$, then Lemma 4 implies that $T \in \mathcal{T}_{max}$. Now let T be a tree obtained from $T' \in \mathcal{T}$ by operation $\mathcal{O}_2$. We have $n^* = n'^* + 2$. Let us observe that $\alpha(T^*) = \alpha(T'^*) + 1$. If $T' \in \mathcal{T}_{max}$, then using Lemma 4 we get $\alpha(T^*) = \alpha(T'^*) + 1 = n'^*/2 + 1 = (n'^* + 2)/2 = n^*/2$. By Lemma 4 we have $T \in \mathcal{T}_{max}$. ■

Now we prove that if the 2-outer-independent domination number of a tree equals the number of leaves plus half of the remaining vertices, then the tree belongs to the family $\mathcal{T}$.

Lemma 6 *Any tree $T \in \mathcal{T}_{max}$ is in $\mathcal{T}$.*

Proof. We prove the result by the induction on the number n of vertices of T. If it has only two vertices, then $T = P_2 \in \mathcal{T}$. Now assume that $n \geq 3$. Assume that the result is true for every tree T' of order $n' < n$.

Assume that some support vertex of T, say x, has degree at least three. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $T'^* = T^*$. Lemma 4 implies that $T' \in \mathcal{T}_{max}$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T has degree two.

We now root T at a vertex r of maximum eccentricity. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) \geq 3$. Let x be a descendant of u other than v. Since every support vertex of T has degree two, the vertex x is not a leaf. Thus it is a support vertex. Let $T' = T - T_v$. Let us observe that $n'^* = n^* - 1$ and $\alpha(T'^*) = \alpha(T^*) - 1$. Using Lemma 4 we get $\alpha(T'^*) = \alpha(T^*) - 1 = n^*/2 - 1 = (n'^* + 1)/2 - 1 = n'^*/2 - 1/2 < n'^*/2$. This is a contradiction as T'^* is bipartite graph.

Now assume that $d_T(u) = 2$. Let $T' = T - T_v$. Let us observe that $n'^* = n^* - 2$ and $\alpha(T'^*) = \alpha(T^*) - 1$. Now we get $\alpha(T'^*) = \alpha(T^*) - 1 = n^*/2 - 1 = (n^* - 2)/2 = n'^*/2$. Lemma 4 implies that $T' \in \mathcal{T}_{max}$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. ■

As a consequence of Lemmas 3, 5 and 6 we get the final result, an upper bound on the 2-outer-independent domination number of a tree together with the characterization of the extremal trees.

Theorem 7 *If T is a nontrivial tree of order n with l leaves, then $\gamma_2^{oi}(T) \le (n+l)/2$ with equality if and only if $T \in \mathcal{T}$.*

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