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A modified hat problem

Abstract. The topic of our paper is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win. There are known many variations of the hat problem. In this paper we consider a variation in which there are $n \geq 3$ players, and blue and red hats. Players do not have to guess their hat colors simultaneously. In this variation of the hat problem players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence "I have got a red hat" or "I have got a blue hat". If someone wants to resign from answering, then he does not do anything. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. Is there a strategy such that the team always succeeds? We give an optimal strategy for the problem which always succeeds. Additionally, we prove in which step the team wins using the strategy. We also prove what is the greatest possible number of steps that are needed for the team to win using the strategy.

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1. Introduction. In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [21], Die Zeit [6], and abcNews [20]. It is also a one of subjects of the webpage [4].

The hat problem with $2^k - 1$ players was solved in [14], and for 2^k players in [11]. The problem with n players was investigated in [7]. The hat problem and Hamming codes were the subject of [8].

There are known many variations of the hat problem. For example the generalized hat problem with n people and q colors was investigated in [19]. In the papers [1, 10, 18] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [16] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing desired number of correct guesses. In [17] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [5], linear programming [16], genetic programming [9], economy [1, 18], biology [17], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15].

In this paper we consider a variation in which there are $n \geq 3$ players, and blue and red hats. Players do not have to guess their hat colors simultaneously. In this variation of the hat problem players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence “I have got a red hat” or “I have got a blue hat”. If someone wants to resign from answering, then he does not do anything. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. Is there a strategy such that the team always succeeds? We give an optimal strategy for the problem which always succeeds. Additionally, we prove in which step the team wins using the strategy. We also prove what is the greatest possible number of steps that are needed for the team to win using the strategy.

2. Modified hat problem. Let us consider a modified hat problem which we define as follows. There are $n \geq 3$ players and two colors (red and blue) in which players do not have to guess their hat colors simultaneously. Players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence “I have got a red hat” or “I have got a blue hat”. If someone wants to resign from answering, then he does not do anything. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. Is there a strategy such that everybody wins?

We give an optimal strategy for the problem which always succeeds. Additionally, we prove in which step the team wins using the strategy. We also prove what is the greatest possible number of steps that are needed for the team to win using the strategy.

Let us consider the following strategy for the modified hat problem.

Strategy 1 Players proceed as follows.

Step 1 (one minute after the beginning)
Only these players who see the hats of one color only come to the basket. There are three possibilities:

- Only one player comes to the basket. Then he guesses he has a hat of the color differing from the one he sees.

- More than one player come to the basket. Then every one of them guesses he has a hat of the color which he sees.
- No player comes to the basket. Then we execute Step 2.

Let i be a positive integer.

Step $2i$ ($2i$ minutes after the beginning)
Only these players who see exactly i blue hats come to the basket. There are two possibilities:

- At least one player comes to the basket. Then every one of them guesses he has a blue hat.
- No player comes to the basket. Then we execute Step $2i+1$.

Step $2i+1$ ($(2i+1)$ minutes after the beginning)
Only these players who see exactly i red hats come to the basket. There are two possibilities:

- At least one player comes to the basket. Then every one of them guesses he has a red hat.
- No player comes to the basket. Then we execute Step $2i+2$.

Now we prove that this strategy always succeeds.

THEOREM 2.1 *Strategy 1 always succeeds for the modified hat problem.*

PROOF If all players have hats of the same color, then in Step 1 every player guesses his hat color correctly. Thus the team wins. If one player has a hat of some color, while the remaining $n-1$ players have hats of another color, then in Step 1 only the player which has a hat of the unique color guesses its color, and the guess is correct. Therefore the team wins.

If there are $n = 3$ players, then the team wins in Step 1, as there is no other possibility.

Now assume that there are $n \geq 4$ players, and at least two of them have red hats and at least two of them have blue hats. Let i be a positive integer. Now we prove that if i is odd (even, respectively), then if in the executed Step i no player comes to the basket, then every player sees at least $(i+1)/2$ red hats ($(i/2+1)$ blue hats, respectively). We prove that by induction. First, assume that $i = 1$. Since no player sees hats only of the one color (as no player comes to the basket in Step 1), every player sees at least one hat of each color. Now assume that $i = 2$. Since every player sees at least one blue hat (as no player has come to the basket in Step 1) and no player sees exactly one blue hat (as no player comes to the basket in Step 2), it follows that every player sees at least two blue hats. Let k be a positive integer. Assume that if no player comes to the basket in Step $2k-1$, then every player sees at least k red hats, and if no player comes to the basket in Step $2k$, then every player sees at least $k+1$ blue hats. First, we prove that if in the executed Step $2k+1$ no player comes to the basket, then every player sees at least $k+1$ red hats. Since every player sees at least k red hats (as no player has come to the basket in Step $2k-1$) and no player sees exactly k red hats (as no player comes to the basket in Step $2k+1$), it follows that every player sees at least $k+1$ red hats. Now, we prove that if in the executed Step $2k+2$ no player comes to the basket, then every player sees at least $k+2$ blue hats. Since every player sees at least $k+1$ blue hats (as no player has come to the basket in Step $2k$) and no player sees exactly $k+1$ blue hats (as no player comes to the basket in Step $2k+2$), it follows that every player sees at least $k+2$ blue hats.

Now we prove that if some player guesses his hat color in any Step i, then his guess is correct. First assume that $i=2$. Since every player sees at least one blue hat (as no player has come to the basket in Step 1), there are at least two blue hats (as particularly the player who has a blue hat also sees at least one blue hat). Some player comes to the basket in Step 2, thus he sees exactly one blue hat. This implies that he has a blue hat, and therefore his guess that he has a blue hat is correct. Now assume that $i \geq 4$ is an even integer, that is $i = 2k+2$, where k is a positive integer. First, assume that some player comes to the basket in the executed Step $2k+2$. No player has come to the basket in Steps $2k$ and $2k+1$, thus every player sees at least $k+1$ blue hats and at least $k+1$ red hats. Since every player who has a blue hat sees at least $k+1$ blue hats, there are at least $k+2$ blue hats. The person who comes to the basket in Step $2k+2$ sees exactly $k+1$ blue hats. This implies that he has a blue hat, and therefore his guess that he has a blue hat is correct. Now, assume that some player comes to the basket in the executed Step $2k+1$. Since every player sees at least k red hats (as no player has come to the basket in Step $2k-1$), there are at least $k+1$ red hats. The person who comes to the basket in Step $2k+1$ sees exactly k red hats. This implies that he has a red hat, and therefore his guess that he has a red hat is correct. ■

Let us consider the numbers of red and blue hats on the heads of players. If there are less red hats than blue hats, then let x mean the number of red hats. Otherwise let it mean the number of blue hats.

Now we prove in which step the team wins using Strategy 1.

FACT 2.2 *If there are less than two blue hats or less than two red hats, then Strategy 1 succeeds in Step* 1. *Otherwise, for x defined above, Strategy 1 succeeds in Step $2x-1$ if there are more blue hats than red hats, and otherwise in Step $2x-2$.*

PROOF If there are less than two blue hats or less than two red hats, then from the proof of Theorem 2.1 we know that the team wins in Step 1. Now assume that there are at least two hats of each color. First, let us assume that there are more blue hats than red hats. Thus there are exactly x red hats and more than x blue hats. From the proof of Theorem 2.1 we know that some player having a red (blue, respectively) hat would guess his hat color correctly in Step $2x-1$ ($2x$ or further, respectively). This implies that the team wins in Step $2x-1$. Now, assume that the number of blue hats is smaller than or equal to the number of red hats. Thus there are exactly x blue hats and at least x red hats. From the proof of Theorem 2.1

we know that some player having a blue (red, respectively) hat would guess his hat color correctly in Step $2x-2$ ($2x-1$ or further, respectively). This implies that the team wins in Step $2x-2$. ■

Now we prove what is the greatest possible number of steps that are needed for the team to win using Strategy 1.

COROLLARY 2.3 *The greatest possible number of step in which the team wins using Strategy 1 is* $n-2$.

PROOF Let n mean the number of players. From Fact 2.2 we know that if there are less than two blue hats or less than two red hats, then the team wins in Step 1. Since $n \geq 3$, we have $n-2 \geq 1$. Now assume that there are at least two blue hats and at least two red hats. If there are more blue hats than red hats, then obviously $x < n/2$, that is, $x \leq (n-1)/2$. By Fact 2.2, the team wins in Step $2x-1$. We have $2x-1 \leq n-1-1 = n-2$. Now assume that the number of blue hats is smaller than or equal to the number of red hats. Obviously, $x \leq n/2$. By Fact 2.2, the team wins in Step $2x-2$. We have $2x-2 \leq n-2$. ■

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