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On the ratio between 2-domination and total outer-independent domination numbers of trees*

Marcin KRZYWKOWSKI**

Abstract A 2-dominating set of a graph G is a set D of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D. A total outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The 2-domination (total outer-independent domination, respectively) number of a graph G is the minimum cardinality of a 2-dominating (total outer-independent dominating, respectively) set of G. We investigate the ratio between 2-domination and total outer-independent domination numbers of trees.

Keywords 2-domination, total domination, total outer-independent domination, tree
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The path on n vertices we denote by P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set, abbreviated 2DS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. A 2-dominating set of G of minimum cardinality is called a $\gamma_2(G)$-set. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple

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**Department of Algorithms and System Modelling, Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology
Narutowicza 11/12, 80-233 Gdańsk, Poland
E-mail: marcin.krzywkowski@gmail.com
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domination was introduced by Fink and Jacobson [12], and further studied for example in [4, 5, 14, 15, 20, 22]. For a comprehensive survey of domination in graphs, see [16, 17].

A subset $D \subseteq V(G)$ is a total dominating set of G if every vertex of G has a neighbor in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [7], and further studied for example in [1–3, 8–11, 13, 18, 19, 23, 24].

A subset $D \subseteq V(G)$ is a total outer-independent dominating set, abbreviated TOIDS, of G if every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. A total outer-independent dominating set of G of minimum cardinality is called a $\gamma_t^{oi}(G)$-set. The study of total outer-independent domination in graphs was initiated in [21].

The authors of [6] gave upper bounds on the ratios of several domination parameters in trees.

We investigate the ratio between 2-domination and total outer-independent domination numbers of trees.

2 Results

Since the one-vertex graph does not have a total outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following three straightforward observations.

Observation 2.1 Every support vertex of a graph G is in every $\gamma_t^{oi}(G)$-set.

Observation 2.2 For every connected graph G of diameter at least three there exists a $\gamma_t^{oi}(G)$-set that contains no leaf.

Observation 2.3 Every leaf of a graph G is in every $\gamma_2(G)$-set.

Let T be a tree. Let us observe that the ratio $\gamma_2(T)/\gamma_t^{oi}(T)$ is not bounded above as attaching a new vertex to any support vertex increases the 2-domination number while not affecting the total outer-independent domination number.

We show that $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \geq 3/4$, for every tree T. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a path P_6 by joining one of its leaves to a vertex of T_k adjacent to a path P_6.
- Operation $\mathcal{O}_2$: Attach a path P_6 by joining one of its leaves to a vertex of T_k adjacent to a support vertex of degree two.
- Operation $\mathcal{O}_3$: Attach a path P_6 by joining one of its leaves to any leaf of $T_k \neq P_2$.

Now we prove that for every tree of the family $\mathcal{T}$, the ratio between the 2-domination number plus one and the total outer-independent domination number plus two equals three fourths.

Lemma 2.1 *If $T \in \mathcal{T}$, then $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = 3/4$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 \in \{P_2, P_3\}$, then $\gamma_2(T) = \gamma_t^{oi}(T) = 2$. We have $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = 3/4$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' operation $\mathcal{O}_1$. The vertex to which is attached P_6 we denote by x. Let $v_1v_2v_3v_4v_5v_6$ be the attached path. Let v_1 be joined to x. Let $abcdef$ denote a path P_6 adjacent to x and different from $v_1v_2v_3v_4v_5v_6$. Let x and a be adjacent. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertices d, b, and x. Let D' be such a set. It is easy to observe that $D' \cup \{v_2, v_4, v_6\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v_5, v_3, and v_1. Let D be such a set. By Observation 2.3 we have $v_6 \in D$. No one of the vertices v_2 and v_4 has a neighbor in D, thus $v_2, v_4 \in D$. Observe that $D \setminus \{v_2, v_4, v_6\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 3$. This implies that $\gamma_2(T) = \gamma_2(T') + 3$. Now let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertices v_3 and c, and does not contain any leaf. Let D be such a set. By Observation 2.1 we have $v_5 \in D$. Each one of the vertices b, v_2, and v_5 has to have a neighbor in D, thus $a, v_1, v_4 \in D$. We have $v_2 \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1, v_2, v_4, v_5\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 4$. This implies that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 4$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = (\gamma_2(T')+1+3)/(\gamma_t^{oi}(T')+6)$ $= (3(\gamma_t^{oi}(T')+2)/4+3)/(\gamma_t^{oi}(T')+6) = (3\gamma_t^{oi}(T')/4+9/2)/(\gamma_t^{oi}(T')+6) = 3/4$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which is attached P_6 we denote by x. Let $v_1v_2v_3v_4v_5v_6$ be the attached path. Let v_1 be joined to x. Let a denote a support vertex of degree two adjacent to x. The leaf adjacent to a we denote by b. Let us observe that there exists a $\gamma_2(T')$-set that contains the vertex x. Let D' be such a set. It is easy to observe that $D' \cup \{v_2, v_4, v_6\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 3$. In the same way as when considering the operation $\mathcal{O}_1$ we conclude that $\gamma_2(T') \leq \gamma_2(T) - 3$. This implies that $\gamma_2(T) = \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex v_3, and does not contain any leaf. Let D be such a set. By Observation 2.1 we have $v_5, a \in D$. Each one of the vertices v_2 and v_5 has to have a neighbor in D, thus $v_1, v_4 \in D$. We have $v_2 \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1, v_2, v_4, v_5\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 4$. In the same way as when considering the operation $\mathcal{O}_1$ we conclude that $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$. This implies that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 4$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = (\gamma_2(T')+1+3)/(\gamma_t^{oi}(T')+6) = (3(\gamma_t^{oi}(T')+2)/4+3)/(\gamma_t^{oi}(T')+6)$ $= (3\gamma_t^{oi}(T')/4+9/2)/(\gamma_t^{oi}(T')+6) = 3/4$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. The leaf to which is attached P_6 we denote by x. Let $v_1v_2v_3v_4v_5v_6$ be the attached path. Let v_1 be joined to x. The neighbor of x other than v_1 we denote by y. Let D' be any $\gamma_2(T')$-set. By Observation 2.3 we have $x \in D'$. It is easy to observe that $D' \cup \{v_2, v_4, v_6\}$ is a 2DS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T') + 3$. In the same way as when considering the operation $\mathcal{O}_1$ we conclude that $\gamma_2(T') \leq \gamma_2(T) - 3$. This implies that $\gamma_2(T) = \gamma_2(T') + 3$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertices v_6, v_3, and x. Let D be such a set. By Observation 2.1 we have $v_5 \in D$. Each one of the vertices v_2 and v_5 has to have a neighbor in D, thus $v_1, v_4 \in D$. We have $v_2, y \in D$ as the set $V(T) \setminus D$ is independent. Let us observe that $D \setminus \{v_1, v_2, v_4, v_5\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1, v_2, v_4, v_5\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 4$. In the same way as when considering the operation $\mathcal{O}_1$ we conclude that $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 4$. This implies that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 4$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = (\gamma_2(T')+1+3)/(\gamma_t^{oi}(T')+6)$ $= (3(\gamma_t^{oi}(T')+2)/4+3)/(\gamma_t^{oi}(T')+6) = (3\gamma_t^{oi}(T')/4+9/2)/(\gamma_t^{oi}(T')+6) = 3/4$. □

Now we establish the main result, a lower bound on the ratio between the 2-domination number of a tree plus one and its total outer-independent domination number plus two, together with a characterization of the extremal trees.

Theorem 2.1 *If T is a tree, then $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \geq 3/4$ with equality if and only if $T \in \mathcal{T}$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. By Lemma 2.1 we have $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2)$ $= 3/4$. Now assume that $\mathrm{diam}(T) = 2$. Thus T is a star. We have $\gamma_2(T) = n-1$ and $\gamma_t^{oi}(T) = 2$. We get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = n/4 \geq 3/4$. If $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) = 3/4$, then $n = 3$. Consequently, $T = P_3 \in \mathcal{T}$.

Now assume that $\mathrm{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is adjacent to at least three leaves. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_t^{oi}(T')$-set. By Observation 2.1 we have $x \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')$. Now let D be any $\gamma_2(T)$-set. By Observation 2.3 we have $y \in D$. Let us observe that $D \setminus \{y\}$ is a 2DS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{y\}$. Therefore $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \geq (\gamma_2(T')+2)/(\gamma_t^{oi}(T')+2) > (\gamma_2(T')+1)/(\gamma_t^{oi}(T')+2) \geq 3/4$. Henceforth, we can assume that every support vertex of T is adjacent to at most two leaves.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $\mathrm{diam}(T) \geq 4$, then let w be the parent of u. If $\mathrm{diam}(T) \geq 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geq 6$, then let e be the parent of d. If $\mathrm{diam}(T) \geq 7$, then let f be the parent of e. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(v) = 3$. The leaf adjacent to v and different from t we denote by a. Let $T' = T - T_v$. Let D' be any $\gamma_t^{oi}(T')$-set. If $u \in D'$, then it is easy to see that $D' \cup \{v\}$ is a TOIDS of the tree T. Now assume that $u \notin D$. Let us observe that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t, a \in D$. Observe that $D \setminus \{t, a\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \geq (\gamma_2(T')+1+2)/(\gamma_t^{oi}(T')+4) \geq (3(\gamma_t^{oi}(T')+2)/4+2)/(\gamma_t^{oi}(T')+4)$ $= (3\gamma_t^{oi}(T')/4+7/2)/(\gamma_t^{oi}(T')+4) = (3/4) \cdot (\gamma_t^{oi}(T')+14/3)/(\gamma_t^{oi}(T')+4) > 3/4$.

Now assume that $d_T(v)=2$. First assume that among the children of u there is a support vertex, say x, different from v. Let $T'=T-T_v$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex x has to have a neighbor in D', thus $u\in D'$. It is easy to see that $D'\cup\{v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T)\le\gamma_t^{oi}(T')+1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t\in D$. Observe that $D\setminus\{t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T')\le\gamma_2(T)-1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2)\ge(\gamma_2(T')+1+1)/(\gamma_t^{oi}(T')+3)\ge(3(\gamma_t^{oi}(T')+2)/4+1)/(\gamma_t^{oi}(T')+3)$ $=(3\gamma_t^{oi}(T')/4+5/2)/(\gamma_t^{oi}(T')+3)=(3/4)\cdot(\gamma_t^{oi}(T')+10/3)/(\gamma_t^{oi}(T')+3)>3/4$.

Now assume that some child of u, say x, is a leaf. Let $T'=T-x$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex v has to have a neighbor in D', thus $u\in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T)\le\gamma_t^{oi}(T')$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. By Observation 2.3 we have $x\in D$. It is easy to observe that $D\setminus\{x\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T')\le\gamma_2(T)-1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2)\ge(\gamma_2(T')+2)/(\gamma_t^{oi}(T')+2)>(\gamma_2(T')+1)/(\gamma_t^{oi}(T')+2)\ge 3/4$.

Now assume that $d_T(u)=2$. Let k be a child of w different from u. Let us observe that it suffices to consider only the possibilities when $d_T(k)\le 2$. First assume that $d_T(w)\ge 4$. Let l be a child of w different from u and k. Let us observe that it suffices to consider only the possibilities when $d_T(l)\le 2$. Let $T'=T-T_u$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D'\cup\{u,v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T)\le\gamma_t^{oi}(T')+2$. Let us observe that there exists a $\gamma_2(T)$-set that contains the vertex u. Let D be such a set. By Observation 2.3 we have $t\in D$. The set D is minimal, thus $v\notin D$. If $w\in D$, then it is easy to observe that $D\setminus\{u,t\}$ is a 2DS of the tree T'. Now assume that $w\notin D$. Thus both vertices k and l belong to the set D as each one of them has at most one neighbor in the set D. Let us observe that $D\setminus\{u,t\}$ is a 2DS of the tree T' as the vertex w has at least two neighbors in $D\setminus\{u,t\}$. Therefore $\gamma_2(T')\le\gamma_2(T)-2$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2)\ge(\gamma_2(T')+1+2)/(\gamma_t^{oi}(T')+4)\ge(3(\gamma_t^{oi}(T')+2)/4+2)/(\gamma_t^{oi}(T')+4)$ $=(3\gamma_t^{oi}(T')/4+7/2)/(\gamma_t^{oi}(T')+4)=(3/4)\cdot(\gamma_t^{oi}(T')+14/3)/(\gamma_t^{oi}(T')+4)>3/4$.

Now assume that $d_T(w)=3$. First assume that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possiility when T_k is a path P_3, say klm. Let $T'=T-T_w$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D'\cup\{w,u,v,k,l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T)\le\gamma_t^{oi}(T')+5$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, l, and w. Let D be such a set. By Observation 2.3 we have $t,m\in D$. No one of the vertices u and k has a neighbor in the set D, thus $u,k\in D$. Observe that $D\setminus\{u,t,k,m\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T')\le\gamma_2(T)-4$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2)\ge(\gamma_2(T')+1+4)/(\gamma_t^{oi}(T')+7)\ge(3(\gamma_t^{oi}(T')+2)/4+4)/(\gamma_t^{oi}(T')+7)$ $=(3\gamma_t^{oi}(T')/4+11/2)/(\gamma_t^{oi}(T')+7)=(3/4)\cdot(\gamma_t^{oi}(T')+22/3)/(\gamma_t^{oi}(T')+7)>3/4$.

Now assume that the distance of w to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T'=T-T_u$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D'\cup\{u,v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T)\le\gamma_t^{oi}(T')+2$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u and w. Let D be such a set. By Observation 2.3 we have $t\in D$. The set D is minimal, thus $v\notin D$. It is easy to observe that $D\setminus\{u,t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T')\le\gamma_2(T)-2$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2)\ge(\gamma_2(T')+1+2)/(\gamma_t^{oi}(T')+4)\ge(3(\gamma_t^{oi}(T')+2)/4$ $+2)/(\gamma_t^{oi}(T')+4)=(3\gamma_t^{oi}(T')/4+7/2)/(\gamma_t^{oi}(T')+4)=(3/4)\cdot(\gamma_t^{oi}(T')+14/3)/(\gamma_t^{oi}(T')+4)>3/4$.

Now assume that k is a leaf. Let $T' = T - T_w$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{w, u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 3$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v and w. Let D be such a set. By Observation 2.3 we have $t, k \in D$. The vertex u has no neighbor in the set D, thus $u \in D$. Observe that $D \setminus \{u, t, k\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 3$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \geq (\gamma_2(T')+1+3)/(\gamma_t^{oi}(T')+5) \geq (3(\gamma_t^{oi}(T')+2)/4+3)/(\gamma_t^{oi}(T')+5)$ $= (3\gamma_t^{oi}(T')/4 + 9/2)/(\gamma_t^{oi}(T') + 5) = (3/4) \cdot (\gamma_t^{oi}(T') + 6)/(\gamma_t^{oi}(T') + 5) > 3/4$.

If $d_T(w) = 1$, then $T = P_4$. We get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) = 4/4 > 3/4$, a contradiction. Now assume that $d_T(w) = 2$. First assume that there is a child of d other than w, say k, such that the distance of d to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_w$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex k. Let D' be such as set. The set $V(T') \setminus D'$ is independent, thus $d \in D'$. It is easy to see that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v and w. Let D be such a set. By Observation 2.3 we have $t \in D$. The vertex u has no neighbor in D, thus $u \in D$. Observe that $D \setminus \{u, t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 2$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \geq (\gamma_2(T') + 1 +2)/(\gamma_t^{oi}(T') + 4) \geq (3(\gamma_t^{oi}(T') + 2)/4 + 2)/(\gamma_t^{oi}(T') + 4) = (3\gamma_t^{oi}(T')/4 + 7/2)/(\gamma_t^{oi}(T') + 4)$ $= (3/4) \cdot (\gamma_t^{oi}(T') + 14/3)/(\gamma_t^{oi}(T') + 4) > 3/4$.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_v$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $w \in D'$. Each one of the vertices w and l has to have a neighbor in D', thus $d, k \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \geq (\gamma_2(T') + 1 +1)/(\gamma_t^{oi}(T') + 3) \geq (3(\gamma_t^{oi}(T') + 2)/4 + 1)/(\gamma_t^{oi}(T') + 3) = (3\gamma_t^{oi}(T')/4 + 5/2)/(\gamma_t^{oi}(T') + 3)$ $= (3/4) \cdot (\gamma_t^{oi}(T') + 10/3)/(\gamma_t^{oi}(T') + 3) > 3/4$.

Now assume that there is a child of d, say k, such that the distance of d to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The set $V(T') \setminus D'$ is independent, thus $d \in D'$. It is easy to see that $D' \cup \{k\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')+1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex k. Let D be such a set. By Observation 2.3 we have $l \in D$. Observe that $D \setminus \{l\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \leq \gamma_2(T) - 1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \geq (\gamma_2(T')+1+1)/(\gamma_t^{oi}(T')+3) \geq (3(\gamma_t^{oi}(T')+2)/4+1)/(\gamma_t^{oi}(T')+3)$ $= (3\gamma_t^{oi}(T')/4 + 5/2)/(\gamma_t^{oi}(T') + 3) = (3/4) \cdot (\gamma_t^{oi}(T') + 10/3)/(\gamma_t^{oi}(T') + 3) > 3/4$.

Now assume that some child of d, say k, is a leaf. Let $T' = T - k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The set $V(T') \setminus D'$ is independent, thus $d \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u and d. Let D be such a set. By Observation 2.3 we have $k \in D$. It is easy to observe that $D \setminus \{k\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \ge (\gamma_2(T')+2)/(\gamma_t^{oi}(T')+2) > (\gamma_2(T')+1)/(\gamma_t^{oi}(T')+2) \ge 3/4$.

Now assume that $d_T(d) = 2$. First assume that there is a child of e other than d, say k, such that the distance of e to the most distant vertex of T_k is five. It suffices to consider only the possibility when T_k is a path P_5, say $klmpq$. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex l, and does not contain any leaf. Let D' be such a set. By Observation 2.1 we have $w \in D'$. Each one of the vertices w and k has to have a neighbor in D', thus $d, e \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \ge (\gamma_2(T')+1+1)/(\gamma_t^{oi}(T')+3) \ge (3(\gamma_t^{oi}(T')+2)/4+1)/(\gamma_t^{oi}(T')+3)$ $= (3\gamma_t^{oi}(T')/4+5/2)/(\gamma_t^{oi}(T')+3) = (3/4)\cdot(\gamma_t^{oi}(T')+10/3)/(\gamma_t^{oi}(T')+3) > 3/4$.

Now assume that there is a child of e, say k, such that the distance of e to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The vertex d has to have a neighbor in D', thus $e \in D'$. It is easy to see that $D' \cup \{l, m\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices m and k. Let D be such a set. By Observation 2.3 we have $p \in D$. The vertex l has no neighbor in the set D, thus $l \in D$. Observe that $D \setminus \{l, p\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 2$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \ge (\gamma_2(T')+1+2)/(\gamma_t^{oi}(T')+4) \ge (3(\gamma_t^{oi}(T')+2)/4+2)/(\gamma_t^{oi}(T')+4)$ $= (3\gamma_t^{oi}(T')/4+7/2)/(\gamma_t^{oi}(T')+4) = (3/4)\cdot(\gamma_t^{oi}(T')+14/3)/(\gamma_t^{oi}(T')+4) > 3/4$.

Now assume that there is a child of e, say k, such that the distance of e to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex w. Let D' be such a set. The vertex d has to have a neighbor in D', thus $e \in D'$. It is easy to see that $D' \cup \{k\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex k. Let D be such a set. By Observation 2.3 we have $l \in D$. Observe that $D \setminus \{l\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \ge (\gamma_2(T')+1+1)/(\gamma_t^{oi}(T')+3)$ $\ge (3(\gamma_t^{oi}(T')+2)/4+1)/(\gamma_t^{oi}(T')+3) = (3\gamma_t^{oi}(T')/4+5/2)/(\gamma_t^{oi}(T')+3) = (3/4)\cdot(\gamma_t^{oi}(T')$ $+10/3)/(\gamma_t^{oi}(T')+3) > 3/4$.

Now assume that some child of e, say k, is a leaf. Let $T' = T - T_v$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $w, e \in D'$. The vertex w has to have a neighbor in D', thus $d \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex v. Let D be such a set. By Observation 2.3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \ge (\gamma_2(T')+1+1)/(\gamma_t^{oi}(T')+3) \ge (3(\gamma_t^{oi}(T')+2)/4+1)/(\gamma_t^{oi}(T')+3)$ $= (3\gamma_t^{oi}(T')/4+5/2)/(\gamma_t^{oi}(T')+3) = (3/4)\cdot(\gamma_t^{oi}(T')+10/3)/(\gamma_t^{oi}(T')+3) > 3/4$.

Now assume that the distance of e to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. First assume that $d_T(e) \ge 4$. Let p

denote a child of e different from d and k. It suffices to consider only the possibility when T_p is a path P_3, say pqs. Let $T' = T - T_k$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{k, l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi} \le \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, k, and p. Let D be such a set. By Observation 2.3 we have $m \in D$. The set D is minimal, thus $l \notin D$. Let us observe that $D \setminus \{k, m\}$ is a 2DS of the tree T' as the vertex e has at least two neighbors in $D \setminus \{k, m\}$. Therefore $\gamma_2(T') \le \gamma_2(T) - 2$. Now we get $(\gamma_2(T)+1)/(\gamma_t^{oi}(T)+2) \ge (\gamma_2(T')+1+2)/(\gamma_t^{oi}(T')+4) \ge (3(\gamma_t^{oi}(T')+2)/4+2)/(\gamma_t^{oi}(T')+4)$ $= (3\gamma_t^{oi}(T')/4 + 7/2)/(\gamma_t^{oi}(T') + 4) = (3/4) \cdot (\gamma_t^{oi}(T') + 14/3)/(\gamma_t^{oi}(T') + 4) > 3/4$.

Now assume that $d_T(e) = 3$. Let $T' = T - T_e$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{e, d, u, v, k, l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 6$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, w, l, and e. Let D be such a set. By Observation 2.3 we have $t, m \in D$. No one of the vertices u, d, and k has a neighbor in the set D, thus $u, d, k \in D$. Observe that $D \setminus \{d, u, t, k, m\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 5$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1$ $+5)/(\gamma_t^{oi}(T') + 8) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 5)/(\gamma_t^{oi}(T') + 8) = (3\gamma_t^{oi}(T')/4 + 13/2)/(\gamma_t^{oi}(T') + 8)$ $= (3/4) \cdot (\gamma_t^{oi}(T') + 26/3)/(\gamma_t^{oi}(T') + 8) > 3/4$.

Now assume that $d_T(e) = 2$. First assume that there is a child of f other than e, say k, such that the distance of f to the most distant vertex of T_k is six. It suffices to consider only the possibility when T_k is a path P_6, say $klmpqs$. Let $T' = T - T_e$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to observe that $D' \cup \{e, d, u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, w, and e. Let D be such a set. By Observation 2.3 we have $t \in D$. No one of the vertices u and d has a neighbor in the set D, thus $u, d \in D$. Observe that $D \setminus \{d, u, t\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 3$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1 + 3)/(\gamma_t^{oi}(T') + 6)$ $\ge (3(\gamma_t^{oi}(T') + 2)/4 + 3)/(\gamma_t^{oi}(T') + 6) = (3\gamma_t^{oi}(T')/4 + 9/2)/(\gamma_t^{oi}(T') + 6) = 3/4$. If $(\gamma_2(T)$ $+1)/(\gamma_t^{oi}(T) + 2) = 3/4$, then $(\gamma_2(T') + 1)/(\gamma_t^{oi}(T') + 2) = 3/4$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is five. It suffices to consider only the possibility when T_k is a path P_5, say $klmpq$. Let $T' = T - T_v - T_k$. Let us observe that there exists a $\gamma_t^{oi}(T')$-set that does not contain the vertex e, and does not contain any leaf. Let D' be such a set. By Observation 2.1 we have $w \in D'$. The vertex w has to have a neighbor in D', thus $d \in D'$. We have $f \in D'$ as the set $V(T') \setminus D'$ is independent. Let us observe that $D' \setminus \{w\} \cup \{e, u, v, k, m, p\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')+5$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, f, m, and k. Let D be such a set. By Observation 2.3 we have $t, q \in D$. The set D is minimal, thus $v, p, l \notin D$. It is easy to observe that $D \setminus \{t, k, m, q\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 4$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1$ $+4)/(\gamma_t^{oi}(T') + 7) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 4)/(\gamma_t^{oi}(T') + 7) = (3\gamma_t^{oi}(T')/4 + 11/2)/(\gamma_t^{oi}(T') + 7)$ $= (3/4) \cdot (\gamma_t^{oi}(T') + 22/3)/(\gamma_t^{oi}(T') + 7) > 3/4$.

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is path P_4, say $klmp$. Let $T' = T - T_w - T_k$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $e \in D'$. The vertex e has to have a neighbor in D', thus $f \in D'$. It is easy to observe that $D' \cup \{d, u, v, l, m\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 5$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertices v, w, m, and k. Let D be such a set. By Observation 2.3 we have $t, p \in D$. No one of the vertices u and l has a neighbor in the set D, thus $u, l \in D$. Observe that $D \setminus \{u, t, l, p\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 4$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1 + 4)/(\gamma_t^{oi}(T') + 7) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 4)/(\gamma_t^{oi}(T') + 7) = (3\gamma_t^{oi}(T')/4 + 11/2)/(\gamma_t^{oi}(T') + 7) = (3/4) \cdot (\gamma_t^{oi}(T') + 22/3)/(\gamma_t^{oi}(T') + 7) > 3/4$.

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_k$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{k, l\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, f, and k. Let D be such a set. By Observation 2.3 we have $m \in D$. The set D is minimal, thus $l \notin D$. It is easy to observe that $D \setminus \{k, m\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 2$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1 + 2)/(\gamma_t^{oi}(T') + 4) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 2)/(\gamma_t^{oi}(T') + 4) = (3\gamma_t^{oi}(T')/4 + 7/2)/(\gamma_t^{oi}(T') + 4) = (3/4) \cdot (\gamma_t^{oi}(T') + 14/3)/(\gamma_t^{oi}(T') + 4) > 3/4$.

Now assume that there is a child of f, say k, such that the distance of f to the most distant vertex of T_k is two. Thus k is a support vertex of degree two. Let $T' = T - T_e$. Similarly as earlier we conclude that $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 4$ and $\gamma_2(T') \le \gamma_2(T) - 3$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1 + 3)/(\gamma_t^{oi}(T') + 6) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 3)/(\gamma_t^{oi}(T') + 6) = (3\gamma_t^{oi}(T')/4 + 9/2)/(\gamma_t^{oi}(T') + 6) = 3/4$. If $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) = 3/4$, then $(\gamma_2(T') + 1)/(\gamma_t^{oi}(T') + 2) = 3/4$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some child of f, say k, is a leaf. Let $T' = T - k$. Let D' be any $\gamma_t^{oi}(T')$-set. It $f \in D'$, then it is easy to see that D' is a TOIDS of the tree T. Now assume that $f \notin D'$. Let us observe that $D' \cup \{f\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_2(T)$-set that contains the vertices u, d, and f. Let D be such a set. By Observation 2.3 we have $k \in D$. It is easy to observe that $D \setminus \{k\}$ is a 2DS of the tree T'. Therefore $\gamma_2(T') \le \gamma_2(T) - 1$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1 + 1)/(\gamma_t^{oi}(T') + 3) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 1)/(\gamma_t^{oi}(T') + 3) = (3\gamma_t^{oi}(T')/4 + 5/2)/(\gamma_t^{oi}(T') + 3) = (3/4) \cdot (\gamma_t^{oi}(T') + 10/3)/(\gamma_t^{oi}(T') + 3) > 3/4$.

Now assume that $d_T(f) = 2$. Let $T' = T - T_e$. Similarly as earlier we conclude that $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') + 4$ and $\gamma_2(T') \le \gamma_2(T) - 3$. Now we get $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) \ge (\gamma_2(T') + 1 + 3)/(\gamma_t^{oi}(T') + 6) \ge (3(\gamma_t^{oi}(T') + 2)/4 + 3)/(\gamma_t^{oi}(T') + 6) = (3\gamma_t^{oi}(T')/4 + 9/2)/(\gamma_t^{oi}(T') + 6) = 3/4$. If $(\gamma_2(T) + 1)/(\gamma_t^{oi}(T) + 2) = 3/4$, then $(\gamma_2(T') + 1)/(\gamma_t^{oi}(T') + 2) = 3/4$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$. $\square$

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