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On trees with double domination number equal to 2-outer-independent domination number plus one

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Abstract A vertex of a graph is said to dominate itself and all of its neighbors. A double dominating set of a graph G is a set D of vertices of G such that every vertex of G is dominated by at least two vertices of D. The double domination number of a graph G is the minimum cardinality of a double dominating set of G. For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a 2-dominating set if every vertex of $V(G) \setminus D$ has at least two neighbors in D, while it is a 2-outer-independent dominating set of G if additionally the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G is the minimum cardinality of a 2-outer-independent dominating set of G. We characterize all trees with double domination number equal to 2-outer-independent domination number plus one.

Keywords double domination, 2-outer-independent domination, 2-domination, tree
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1 Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a subset of $V(G)$ is independent if there is no edge between every two its vertices. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the tree resulting from T by removing the edge vx, and which contains the vertex x, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves. Given trees T_1 and T_2 such that T_2 is an induced subgraph of T_1, by $T_1 - T_2$ we mean the tree obtained from T_1 by removing all vertices of T_2.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a 2-dominating set of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D. The domination (2-domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_2(G)$,

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respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [5], and further studied for example in [2, 3, 6, 7, 12, 14]. For a comprehensive survey of domination in graphs, see [10, 11].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D, and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G. A 2-outer-independent dominating set of G of minimum cardinality is called a $\gamma_2^{oi}(G)$-set. The study of 2-outer-independent domination in graphs was initiated in [13].

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a double dominating set, abbreviated DDS, of G if every vertex of G is dominated by at least two vertices of D. The double domination number of G, denoted by $\gamma_d(G)$, is the minimum cardinality of a double dominating set of G. A double dominating set of G of minimum cardinality is called a $\gamma_d(G)$-set. Double domination in graphs was introduced by Harary and Haynes [9], and further studied for example in [1, 4, 8].

We characterize all trees with double domination number equal to 2-outer-independent domination number plus one.

2 Results

Since the one-vertex graph does not have double dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following three straightforward observations.

Observation 2.1 Every leaf of a graph G is in every $\gamma_2^{oi}(G)$-set.

Observation 2.2 Every leaf of a graph G is in every $\gamma_d(G)$-set.

Observation 2.3 Every support vertex of a graph G is in every $\gamma_d(G)$-set.

It is easy to see that $\gamma_d(P_2) = \gamma_2^{oi}(P_2)$. Now we prove that for every tree different from P_2 the double domination number is greater than the 2-outer-independent domination number.

Lemma 2.1 *For every tree $T \neq P_2$ we have $\gamma_d(T) > \gamma_2^{oi}(T)$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,m}$. We have $\gamma_d(T) = m + 1 > m = \gamma_2^{oi}(T)$. Now assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. We have $\gamma_d(T) = n > n - 1 = \gamma_2^{oi}(T)$.

Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. We will obtain the result by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z mean leaves adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_2^{oi}(T')$-set. Of course, $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $x, y, z \in D$. It is easy to see that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore

$\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T) \ge \gamma_d(T') + 1 > \gamma_2^{oi}(T') + 1 \ge \gamma_2^{oi}(T)$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) = 2$. Let $T' = T - T_v$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $u \in D'$. It is easy to see that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T')+1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v \in D$. Let us observe that $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T) \ge \gamma_d(T') + 1 > \gamma_2^{oi}(T') + 1 \ge \gamma_2^{oi}(T)$.

Now assume that $d_T(u) \ge 3$. First assume that u is adjacent to a leaf, say x. Let $T' = T - T_v$. Let D' be any $\gamma_2^{oi}(T')$-set. Of course, $D' \cup \{v, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t, x, v, u \in D$. It is easy to see that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T) \ge \gamma_d(T') + 2 > \gamma_2^{oi}(T') + 2 \ge \gamma_2^{oi}(T)$.

Now assume that every descendant of u is a support vertex. Let x mean a descendant of u different from v. The leaf adjacent to x we denote by y. Let $T' = T - T_v$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v \in D$. Let us observe that $D \cup \{u\} \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 1$. Now we get $\gamma_d(T) \ge \gamma_d(T') + 1 > \gamma_2^{oi}(T') + 1 \ge \gamma_2^{oi}(T)$. $\square$

We characterize all trees with double domination number equal to 2-outer-independent domination number plus one. For this purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_3, P_4, P_5\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_3 by joining one of its leaves to a vertex of $T_k \neq P_4$ adjacent to a path P_3.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaves to any support vertex of T_k.
- Operation $\mathcal{O}_4$: Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_4.
- Operation $\mathcal{O}_5$: Attach a vertex by joining it to a vertex of T_k adjacent to a path P_4.
- Operation $\mathcal{O}_6$: Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a support vertex of degree two, and to a vertex of degree two the other neighbor of which is a support vertex.

Now we prove that for every tree of the family $\mathcal{T}$, the double domination number is equal to the 2-outer-independent domination number plus one.

Lemma 2.2 *If $T \in \mathcal{T}$, then $\gamma_d(T) = \gamma_2^{oi}(T) + 1$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = P_3$, then obviously $\gamma_d(T) = 3 = 2 + 1 = \gamma_2^{oi}(T) + 1$. If $T = P_4$, then $\gamma_d(T) = 4 = 3 + 1 = \gamma_2^{oi}(T) + 1$. If $T = P_5$, then also $\gamma_d(T) = 4 = 3 + 1 = \gamma_2^{oi}(T) + 1$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. The attached vertex we denote by x, and its neighbor we denote by y. Let D' be any $\gamma_d(T')$-set. By Observation 2.3 we have $y \in D'$. It is easy to see that $D' \cup \{x\}$ is a DDS of the tree T. Thus $\gamma_d(T) \leq \gamma_d(T') + 1$. Now let D be any $\gamma_2^{oi}(T)$-set. By Observation 2.1 we have $x \in D$. If $y \in D$, then it is easy to see that $D \setminus \{x\}$ is a 2OIDS of the tree T'. Now assume that $y \notin D$. Let a and b mean neighbors of y different from x. The set $V(T) \setminus D$ is independent, thus $a, b \in D$. Let us observe that now also $D \setminus \{x\}$ is a 2OIDS of the tree T' as the vertex y has at least two neighbors in $D \setminus \{x\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. Now we get $\gamma_d(T) \leq \gamma_d(T') + 1 = \gamma_2^{oi}(T') + 2 \leq \gamma_2^{oi}(T) + 1$. On the other hand, by Lemma 2.1 we have $\gamma_d^{oi}(T) \geq \gamma_2^{oi}(T) + 1$. This implies that $\gamma_d(T) = \gamma_2^{oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which is attached P_3 we denote by x. Let $v_1v_2v_3$ mean the attached path. Let v_1 be joined to x. The path P_3 adjacent to x and different from $v_1v_2v_3$ we denote by abc. Let a be adjacent to x. Let us observe that there exists a $\gamma_d(T')$-set that does not contain the vertex a. Let D' be such a set. The vertex a has to be dominated twice, thus $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is a DDS of the tree T. Thus $\gamma_d(T) \leq \gamma_d(T') + 2$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that contains the vertex v_1. Let D be such a set. By Observation 2.1 we have $v_3 \in D$. The set D is minimal, thus $v_2 \notin D$. If $x \in D$, then it is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Now assume that $x \notin D$. Let k mean a neighbor of x different from v_1 and a. The set $V(T) \setminus D$ is independent, thus $a, k \in D$. Let us observe that now also $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{v_1, v_3\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. Now we get $\gamma_d(T) \leq \gamma_d(T') + 2 = \gamma_2^{oi}(T) + 3 \leq \gamma_2^{oi}(T) + 1$. This implies that $\gamma_d(T) = \gamma_2^{oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. The vertex to which is attached P_3 we denote by x. Let $v_1v_2v_3$ mean the attached path. Let v_1 be joined to x. Let y mean a leaf adjacent to x. Let D' be any $\gamma_d(T')$-set. By Observation 2.3 we have $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is DDS of the tree T. Thus $\gamma_d(T) \leq \gamma_d(T') + 2$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that contains the vertex v_1. Let D be such a set. By Observation 2.1 we have $v_3, y \in D$. The set D is minimal, thus $v_2 \notin D$. If $x \in D$, then it is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Now assume that $x \notin D$. Let k mean a neighbor of x different from y. The set $V(T) \setminus D$ is independent, thus $k \in D$. Let us observe that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{v_1, v_3\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. Now we get $\gamma_d(T) \leq \gamma_d(T') + 2 = \gamma_2^{oi}(T') + 3 \leq \gamma_2^{oi}(T) + 1$. This implies that $\gamma_d(T) = \gamma_2^{oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_4$. The vertex to which is attached P_3 we denote by x. Let $v_1v_2v_3$ mean the attached path. Let v_1 be joined to x. Let $abcd$ mean a path P_4 adjacent to x. Let x and a be adjacent. Let us observe that there exists a $\gamma_d(T')$-set that does not contain the vertex b. Let D' be such a set. The vertex a has to be dominated twice, thus $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is a DDS of the tree T. Thus $\gamma_d(T) \leq \gamma_d(T') + 2$.

Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that contains the vertices v_1, b, and x. Let D be such a set. By Observation 2.1 we have $v_3 \in D$. The set D is minimal, thus $v_2 \notin D$. It is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. Now we get $\gamma_d(T) \leq \gamma_d(T') + 2 = \gamma_2^{oi}(T') + 3 \leq \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T) = \gamma_2^{oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_5$. Let x mean the attached vertex, and let y mean its neighbor. Let $abcd$ mean a path P_4 adjacent to x. Let x and a be adjacent. Let us observe that there exists a $\gamma_d(T')$-set that does not contain the vertex b. Let D' be such a set. The vertex a has to be dominated twice, thus $x \in D$. It is easy to see that $D' \cup \{y\}$ is a DDS of the tree T. Thus $\gamma_d(T) \leq \gamma_d(T') + 1$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that contains the vertices b and x. Let D be such a set. By Observation 2.1 we have $y \in D$. It is easy to see that $D \setminus \{y\}$ is a 2OIDS of the tree T'. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. Now we get $\gamma_d(T) \leq \gamma_d(T') + 1 = \gamma_2^{oi}(T') + 2 \leq \gamma_2^{oi}(T) + 1$. This implies that $\gamma_d(T) = \gamma_2^{oi}(T) + 1$.

Now assume that T is obtained from T' by operation $\mathcal{O}_6$. The vertex to which is attached P_3 we denote by x. Let $v_1v_2v_3$ mean the attached path. Let v_1 be joined to x. Let y mean a vertex of degree two adjacent to x the other neighbor of which is a support vertex. Let us observe that there exists a $\gamma_d(T')$-set that does not contain the vertex y. Let D' be such a set. The vertex y has to be dominated twice, thus $x \in D'$. It is easy to see that $D' \cup \{v_2, v_3\}$ is a DDS of the tree T. Thus $\gamma_d(T) \leq \gamma_d(T') + 2$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that contains the vertices v_1 and x. Let D be such a set. By Observation 2.1 we have $v_3 \in D$. The set D is minimal, thus $v_2 \notin D$. It is easy to see that $D \setminus \{v_1, v_3\}$ is a 2OIDS of the tree T'. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. Now we have $\gamma_d(T) \leq \gamma_d(T') + 2 = \gamma_2^{oi}(T') + 3 \leq \gamma_2^{oi}(T) + 1$. This implies that $\gamma_d(T) = \gamma_2^{oi}(T) + 1$. □

Now we prove that if the double domination number of a tree is equal to its 2-outer-independent domination number plus one, then the tree belongs to the family $\mathcal{T}$.

Lemma 2.3 *Let T be a tree. If $\gamma_d(T) = \gamma_2^{oi}(T) + 1$, then $T \in \mathcal{T}$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\text{diam}(T) = 2$, then T is a star $K_{1,m}$. If $T = P_3$, then $T \in \mathcal{T}$. If T is a star different from P_3, then it can be obtained from P_3 by a proper number of operations $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Now assume that $\text{diam}(T) = 3$. Thus T is a double star. If $T = P_4$, then $T \in \mathcal{T}$. If T is a double star different from P_4, then T can be obtained from P_4 by proper numbers of operations $\mathcal{O}_1$ performed on the support vertices. Thus $T \in \mathcal{T}$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order of the tree T is an integer $n \geq 5$. The result we obtain by the induction on the number n. Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z mean leaves adjacent to x. Let $T' = T - y$. Let D' be any $\gamma_2^{oi}(T')$-set. Of course, $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $x, y, z \in D$. It is easy to observe that $D \setminus \{y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 1 = \gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. On the other hand, by Lemma 2.1 we have $\gamma_d(T') \geq \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T') = \gamma_2^{oi}(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. If $\mathrm{diam}(T) \geq 5$, then let d be the parent of w. If $\mathrm{diam}(T) \geq 6$, then let e be the parent of d. If $\mathrm{diam}(T) \geq 7$, then let f be the parent of e. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that among the descendants of u there is a support vertex, say x, different from v. The leaf adjacent to x we denote by y. Assume that there exists a $\gamma_d(T)$-set in which the vertex u is dominated at least thrice. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v \in D$. Let $T' = T - T_v$. Let us observe that $D \setminus \{v, t\}$ is a DDS of the tree T' as the vertex u is dominated at least twice. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex u. Let D' be such a set. It is easy to see that $D' \cup \{t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \leq \gamma_2^{oi}(T')$. This is a contradiction as by Lemma 2.1 we have $\gamma_d(T') > \gamma_2^{oi}(T')$. Therefore in every $\gamma_d(T)$-set the vertex u is dominated only twice. This implies that $d_T(u) = 3$ as all leaves and support vertices belong to every $\gamma_d(T)$-set. Let $T'' = T - T_u$. Let D'' be any $\gamma_2^{oi}(T'')$-set. It is easy to observe that $D'' \cup \{u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2(T) \leq \gamma_2(T'') + 3$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. The vertex u is dominated only twice, thus $u \notin D$. Observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 4$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 4 = \gamma_2^{oi}(T) - 3 \leq \gamma_2^{oi}(T')$, a contradiction.

Thus v is the only one support vertex among the descendants of u. Moreover, we have $d_T(u) = 3$. The leaf adjacent to u we denote by x. First assume that there is a descendant of w, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibilities when T_k is isomorphic to T_u, or T_k is a path P_3. First assume that T_k is isomorphic to T_u. The descendant of l which is a support vertex we denote by l. The leaf adjacent to l we denote by m, and the leaf adjacent to k we denote by p. Let $T' = T - T_u - T_l - p$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $k \in D'$. It is easy to observe that $D' \cup \{u, t, x, m, p\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 5$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t, x, m, p, v, u, l, k \in D$. If $w \in D$, then it is easy to observe that $D \setminus \{u, v, t, x, l, m, p\}$ is a DDS of the tree T'. Now assume that $w \notin D$. Let us observe that $D \cup \{w\} \setminus \{u, v, t, x, l, m, p\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 6$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 6 = \gamma_2^{oi}(T) - 5 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that T_k is a path P_3, say klm. Let $T' = T - T_v - x$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $u \in D'$. It is easy to observe that $D' \cup \{t, x\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex k. Let D be such a set. By Observations 2.2 and 2.3 we have $t, x, v, u \in D$. The vertex k has to be dominated twice, thus $w \in D$. It is easy to observe that $D \setminus \{v, t, x\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 3$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 3 = \gamma_2^{oi}(T) - 2 \leq \gamma_2^{oi}(T')$, a contradiction.

Assume that there exists a $\gamma_d(T)$-set in which the vertex w is dominated at least thrice. Let D be such a set. By Observations 2.2 and 2.3 we have $t, x, v, u \in D$. Let $T' = T - T_u$. Let us observe that $D \setminus \{u, v, t, x\}$ is a DDS of the tree T' as the vertex w is dominated at least twice. Therefore $\gamma_d(T') \leq \gamma_d(T) - 4$. Now let D' be any $\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{u,t,x\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 3$. Now we get $\gamma_d(T') \le \gamma_d(T) - 4 = \gamma_2^{oi}(T) - 3 \le \gamma_2^{oi}(T')$, a contradiction. Therefore in every $\gamma_d(T)$-set the vertex w is dominated only twice. This implies that $d_T(w) = 3$ as all leaves and support vertices belong to every $\gamma_d(T)$-set. Moreover, the descendant of w different from u, say k, is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T' = T - T_w$. If $T' = P_2$, then $\gamma_d(T) = 8 = 6+2 = \gamma_2^{oi}(T)+2 > \gamma_2^{oi}(T)+1$, a contradiction. Now assume that $T' \ne P_2$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{w,u,t,x,l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 5$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex w. Let D be such a set. By Observations 2.2 and 2.3 we have $t,x,l,v,u,k \in D$. Observe that $D \setminus \{u,v,t,x,k,l\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 6$. Now we get $\gamma_d(T') \le \gamma_d(T) - 6 = \gamma_2^{oi}(T) - 5 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(u) = 2$. First assume that there is a descendant of w, say x, such that the distance of w to the most distant vertex of T_x is three. It suffices to consider only the possibility when T_k is a path P_3. Let $T' = T - T_u$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to see that $D' \cup \{u,t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t,v \in D$. Observe that $D \setminus \{v,t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \le \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T') = \gamma_2^{oi}(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some descendant of w, say x, is a leaf. Let $T' = T - T_u$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to see that $D' \cup \{u,t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T)$ $\le \gamma_2^{oi}(T') + 2$. Now let D be any $\gamma_d(T)$-set. By Observations 2.2 and 2.3 we have $t,x,v,w \in D$. The set D is minimal, thus $u \notin D$. Observe that $D \setminus \{v,t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \le \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T') = \gamma_2^{oi}(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that there is a descendant of w, say x, such that the distance of w to the most distant vertex of T_x is two. It suffices to consider only the possibility when x is a support vertex of degree two. The leaf adjacent to x we denote by y. First assume that $d_T(w) \ge 4$. Thus there is a descendant of w, say k, which is a support vertex of degree two different from x. Let $T' = T - T_x$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex w. Let D' be such a set. It is easy to see that $D' \cup \{y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T')+1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $x,y,k \in D$. The vertex u has to be dominated twice, thus $w \in D$. It is easy to observe that $D \setminus \{x,y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(w) = 3$. First assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is four. It suffices to consider only the possibilities when T_k is isomorphic to T_w, or T_k is a path P_4. First assume that T_k is isomorphic to T_w. The path P_3 adjacent to k we denote by lmp, and the path P_2 adjacent to k we denote by qs. Let l and q be adjacent to k. Let $T' = T - T_w - T_l - T_q$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $k \in D'$. It is easy to observe that $D' \cup \{w,u,t,y,l,p,s\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 7$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, p, s, v, x, l, q \in D$. Each one of the vertices u and l has to be dominated twice, thus $w, k \in D$. If $d \in D$, then it is easy to observe that $D \setminus \{w, v, t, x, y, m, p, q, s\}$ is a DDS of the tree T'. Now assume that $d \notin D$. Let us observe that $D \cup \{d\} \setminus \{w, v, t, x, y, m, p, q, s\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 8$. Now we get $\gamma_d(T') \le \gamma_d(T) - 8 = \gamma_2^{oi}(T) - 7 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that T_k is a path P_4, say $klmp$. Let $T' = T - T_u - T_x$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $w \in D'$. It is easy to observe that $D' \cup \{u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. Each one of the vertices u and k has to be dominated twice, thus $w, d \in D$. It is easy to observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 4$. Now we get $\gamma_d(T') \le \gamma_d(T) - 4 = \gamma_2^{oi}(T) - 3 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_u - T_x$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $w \in D'$. It is easy to observe that $D' \cup \{u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and k. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. Each one of the vertices u and k has to be dominated twice, thus $w, d \in D$. It is easy to observe that $D \setminus \{v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 4$. Now we get $\gamma_d(T') \le \gamma_d(T) - 4 = \gamma_2^{oi}(T) - 3 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. The leaf adjacent to k we denote by l. First assume that $d_T(d) \ge 4$. Let m mean a descendant of d different from w and k. It suffices to consider only the possibility when m is a support vertex of degree two. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex d. Let D' be such a set. It is easy to see that $D' \cup \{l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $l, k, m \in D$. The vertex m has to be dominated twice, thus $w \in D$. It is easy to observe that $D \setminus \{k, l\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(d) = 3$. Let $T' = T - T_d$. If $T' = P_2$, then $\gamma_d(T) = 9 = 7 + 2 = \gamma_2^{oi}(T) + 2 > \gamma_2^{oi}(T) + 1$, a contradiction. Now assume that $T' \neq P_2$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{d, w, u, t, y, l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 6$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex d. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, l, v, x, k \in D$. The vertex u has to be dominated twice, thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y, k, l\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 7$. Now we get $\gamma_d(T') \le \gamma_d(T) - 7 = \gamma_2^{oi}(T) - 6 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that some descendant of d, say k, is a leaf. Let $T' = T - T_w$. Let D' be any

$\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T')+4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, k, v, x, d \in D$. The vertex u has to be dominated twice, thus $w \in D$. It is easy to observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 5$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 5 = \gamma_2^{oi}(T) - 4 \leq \gamma_2^{oi}(T')$, a contradiction.

We now turn to the possibility $d_T(w) = 2$. First assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_w$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertices l and d. Let D' be such a set. It is easy to observe that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3 we have $t, p, v, m \in D$. Each one of the vertices w and k has to be dominated twice, thus $w, d, k \in D$. It is easy to observe that $D \setminus \{w, v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 3$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 3 = \gamma_2^{oi}(T) - 2 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_k$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to see that $D' \cup \{k, m\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex k. Let D be such a set. By Observations 2.2 and 2.3 we have $m, l \in D$. Observe that $D \setminus \{l, m\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \leq \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T') = \gamma_2^{oi}(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Now assume that there is a descendant of d, say k, such that the distance of d to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. The leaf adjacent to k we denote by l. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertices u and d. Let D' be such a set. It is easy to see that $D' \cup \{l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $l, k \in D$. The vertex w has to be dominated twice, thus $w, d \in D$. It is easy to observe that $D \setminus \{k, l\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that some descendant of d, say k, is a leaf. Let $T' = T - k$. Let D' be any $\gamma_2^{oi}(T')$-set. Of course, $D' \cup \{k\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $k, d \in D$. The vertex w has to be dominated twice, thus $w \in D$. It is easy to observe that $D \setminus \{k\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 1$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 1 = \gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T') = \gamma_2^{oi}(T') + 1$. The tree T can be obtained from T' by operation $\mathcal{O}_5$. Thus $T \in \mathcal{T}$.

If $d_T(d) = 1$, then $T = P_5 \in \mathcal{T}$. We now turn to the possibility $d_T(w) = 3$. Assume that $d_T(d) = 2$. First assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is five. It suffices to consider only the possibilities when T_k is isomorphic to T_d, or T_k is a path P_5. First assume that T_k is isomorphic to T_d. Let l mean the descendant of k. The path P_3 adjacent to l we denote by mpq, and the path P_2 adjacent to l we denote by ab. Let m and a be adjacent to l. Let $T' = T - T_d$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertices m, l, and e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. The vertex w has to be dominated twice, thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 5$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 5 = \gamma_2^{oi}(T) - 4 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that T_k is a path P_5, say $klmpq$. Let $T' = T - T_d - q$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertices l abd e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y, q\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 5$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u, d, and m. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, q, v, x, p \in D$. Each one of the vertices d and l has to be dominated twice, thus $w, e, k, l \in D$. Let us observe that $D \cup \{m\} \setminus \{w, v, t, x, y, l, q\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 6$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 6 = \gamma_2^{oi}(T) - 5 \leq \gamma_d^{oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is four. It suffices to consider only the possibilities when T_k is isomorphic to T_w, or T_k is a path P_4. First assume that T_k is isomorphic to T_w. The path P_3 adjacent to k we denote by lmp, and the path P_2 adjacent to k we denote by qs. Let l and q be adjacent to k. Let $T' = T - T_d - T_l - T_q$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $k \in D'$. If $e \in D'$, then it is easy to observe that $D' \cup \{w, u, t, y, l, p, s\}$ is a 2OIDS of the tree T. Now assume that $e \notin D'$. Let us observe that $D' \cup \{e, w, u, t, y, l, p, s\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 8$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u, d, and l. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, p, s, v, x, m, q \in D$. Each one of the vertices d and l has to be dominated twice, thus $w, e, k \in D$. It is easy to observe that $D \setminus \{w, v, t, x, y, m, p, q, s\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 9$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 9 = \gamma_2^{oi}(T) - 8 \leq \gamma^{oi}(T')$, a contradiction.

Now assume that T_k is a path P_4, say $klmp$. Let $T' = T - T_d$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertices l and e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. The vertex k has to be dominated twice, thus $e, k \in D$. It is easy to observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 5$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 5 = \gamma_2^{oi}(T) - 4 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. Let $T' = T - T_d$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex e. Let D' be such a set. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Therefore $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-

set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. The vertex w has to be dominated twice, thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 5$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 5 = \gamma_2^{oi}(T) - 4 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that some descendant of e, say k, is a leaf. Let $T' = T - T_u$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to see that $D' \cup \{u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v \in D$. Observe that $D \setminus \{v, t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \leq \gamma_2^{oi}(T') + 1$. This implies that $\gamma_d(T') = \gamma_2^{oi}(T') + 1$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_6$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that no descendant of e is a leaf.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let us observe that we can assume that for every descendant of e different from d, say k, the tree T_k is a path P_3. Let $k_1, k_2, \ldots, k_{d_T(e)-2}$ mean the descendants of e different from d. The descendant of k_i we denote by l_i, and the descendant of l_i we denote by m_i. Let $T' = T - T_d - T_{k_1} - T_{k_2} - \ldots - T_{k_{d_T(e)-2}}$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $e \in D'$. It is easy to observe that $D' \cup \{w, u, t, y, k_1, m_1, k_2, m_2, \ldots, k_{d_T(e)-2}, m_{d_T(e)-2}\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2d_T(e)$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices $u, d, k_1, k_2, \ldots, k_{d_T(e)-2}$. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v, y, x, m_1, l_1, m_2, l_2, \ldots, m_{d_T(e)-2}, l_{d_T(e)-2}$. The vertex w has to be dominated twice, thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y, l_1, m_1, l_2, m_2, \ldots, l_{d_T(e)-2}, m_{d_T(e)-2}\}$. Therefore $\gamma_d(T') \leq \gamma_d(T) - 2d_T(e) - 1$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 2d_T(e) - 1 = \gamma_2^{oi}(T) - 2d_T(e) \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(e) = 2$. Let $T' = T - T_d$. If $T' = P_1$, then $\gamma_d(T) = 7 = 5 + 2 = \gamma_2^{oi}(T) + 2 > \gamma_2^{oi}(T) + 1$, a contradiction. If $T' = P_2$, then let $T'' = T - T_u = P_6$. By the inductive hypothesis we have $T'' \in \mathcal{T}$ as $\gamma_d(P_6) = 5 = 4 + 1 = \gamma_2^{oi}(P_6) + 1$. The tree T can be obtained from T'' by operation $\mathcal{O}_6$. Thus $T \in \mathcal{T}$. Now assume that $T' \neq P_1, P_2$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $e \in D'$. It is easy to observe that $D' \cup \{w, u, t, y\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and d. Let D be such a set. By Observations 2.2 and 2.3 we have $t, y, v, x \in D$. The vertex w has to be dominated twice, thus $w \in D$. Observe that $D \setminus \{w, v, t, x, y\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \leq \gamma_d(T) - 5$. Now we get $\gamma_d(T') \leq \gamma_d(T) - 5 = \gamma_2^{oi}(T) - 4 \leq \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(w) = 2$. Assume that $d_T(d) = 2$. First assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is five. It suffices to consider only the possibility when T_k is a path P_5, say $klmpq$. Let $T' = T - T_d - T_l$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $k \in D'$. It is easy to observe that $D' \cup \{d, u, t, m, q\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 5$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and m. Let D be such a set. By Observations 2.2 and 2.3 we have $t, q, v, p \in D$. Each one of the vertices w and l has to be dominated twice, thus $w, d, l, k \in D$. If $e \in D$, then it is easy to observe that $D \setminus \{d, w, v, t, l, p, q\}$ is a DDS of the tree T'. Now assume that $e \notin D$. Let us observe that $D \cup \{e\} \setminus \{d,w,v,t,l,p,q\}$ is DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 6$. Now we get $\gamma_d(T') \le \gamma_d(T) - 6 = \gamma_2^{oi}(T) - 5 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is four. It suffices to consider only the possibility when T_k is a path P_4, say $klmp$. Let $T' = T - T_d$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{d,u,t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and l. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v \in D$. Each one of the vertices w and k has to be dominated twice, thus $w,d,k,e \in D$. It is easy to observe that $D \setminus \{d,w,v,t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 4$. Now we get $\gamma_d(T') \le \gamma_d(T) - 4 = \gamma_2^{oi}(T) - 3 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_w - T_k$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $d \in D'$. It is easy to observe that $D' \cup \{u,t,k,m\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 4$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertices u and k. Let D be such a set. By Observations 2.2 and 2.3 we have $t,m,v,l \in D$. Each one of the vertices w and k has to be dominated twice, thus $w,d,e \in D$. It is easy to observe that $D \setminus \{w,v,t,l,m\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 5$. Now we get $\gamma_d(T') \le \gamma_d(T) - 5 = \gamma_2^{oi}(T) - 4 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that some descendant of e, say k, is a leaf. Let $T' = T - T_w$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $d \in D'$. It is easy to observe that $D' \cup \{u,t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t,k,v,e \in D$. The vertex w has to be dominated twice, thus $w,d \in D$. It is easy to observe that $D \setminus \{w,v,t\}$ is a DDS of the tree T'. Therefore $\gamma_d(T') \le \gamma_d(T) - 3$. Now we get $\gamma_d(T') \le \gamma_d(T) - 3 = \gamma_2^{oi}(T) - 2 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that there is a descendant of e, say k, such that the distance of e to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. The leaf adjacent to k we denote by l. First assume that $d_T(e) \ge 4$. Thus there is a descendant of e, say a, which is a support vertex of degree two, and which is different from k. The leaf adjacent to a we denote by b. Let $T' = T - T_k$. Let us observe that there exists a $\gamma_2^{oi}(T')$-set that contains the vertex e. It is easy to see that $D' \cup \{l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 1$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $l,k,a \in D$. The vertex w is dominated twice, thus $d \in D$. It is easy to observe that $D \setminus \{k,l\}$ is a DDS of the tree T' as the vertex e is still dominated at least twice. Therefore $\gamma_d(T') \le \gamma_d(T) - 2$. Now we get $\gamma_d(T') \le \gamma_d(T) - 2 = \gamma_2^{oi}(T) - 1 \le \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(e) = 3$. Let $T' = T - T_e$. If $T' = P_1$, then $\gamma_d(T) = 8 = 6 + 2 = \gamma_2^{oi}(T)$ $+2 > \gamma_2^{oi}(T) + 1$, a contradiction. If $T' = P_2$, then also $\gamma_d(T) = 8 = 6 + 2 = \gamma_2^{oi}(T) + 2$ $> \gamma_2^{oi}(T) + 1$, a contradiction. Now assume that $T' \ne P_1, P_2$. Let D' be any $\gamma_2^{oi}(T')$-set. It is easy to observe that $D' \cup \{e,d,u,t,l\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \le \gamma_2^{oi}(T') + 5$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t, l, v, k \in D$. The vertex w has to be dominated twice, thus $w, d \in D$. If $e \notin D$, then observe that $D \setminus \{d, w, v, t, k, l\}$ is a DDS of the tree T'. Now assume that $e \in D$. If $f \notin D$, then let us observe that $D \cup \{f\} \setminus \{e, d, w, v, t, k, l\}$ is a DDS of the tree T'. Now assume that $f \in D$. Let z mean a neighbor of f different from e. We have $z \notin D$, otherwise $D \setminus \{e\}$ is a DDS of the tree T, a contradiction to the minimality of D. Let us observe that $D \cup \{z\} \setminus \{e, d, w, v, t, k, l\}$ is a DDS of the tree T'. Now we conclude that $\gamma_d(T') \leq \gamma_d(T) - 6$. We get $\gamma_d(T') \leq \gamma_d(T) - 6 = \gamma_2^{oi}(T) - 5 \leq \gamma_2^{oi}(T')$, a contradiction.

If $d_T(e) = 1$, then $T = P_6$. Let $T' = T - e = P_5 \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_5$. Now assume that $d_T(e) = 2$. Let $T' = T - T_d$. Let D' be any $\gamma_2^{oi}(T')$-set. By Observation 2.1 we have $e \in D'$. It is easy to observe that $D' \cup \{d, u, t\}$ is a 2OIDS of the tree T. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 3$. Now let us observe that there exists a $\gamma_d(T)$-set that does not contain the vertex u. Let D be such a set. By Observations 2.2 and 2.3 we have $t, v \in D$. The vertex w has to be dominated twice, thus $w, d \in D$. If $e \notin D$, then observe that $D \cup \{e\} \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Now assume that $e \in D$. If $f \in D$, then it is easy to see that $D \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Now assume that $f \notin D$. Let us observe that $D \cup \{f\} \setminus \{d, w, v, t\}$ is a DDS of the tree T'. Now we conclude that $\gamma_d(T') \leq \gamma_d(T) - 3$. We get $\gamma_d(T') \leq \gamma_d(T) - 3 = \gamma_2^{oi}(T) - 2 \leq \gamma_2^{oi}(T')$, a contradiction. □

As an immediate consequence of Lemmas 2.2 and 2.3, we have the following characterization of the trees with double domination number equal to 2-outer-independent domination number plus one.

Theorem 2.1 *Let T be a tree. Then $\gamma_d(T) = \gamma_2^{oi}(T) + 1$ if and only if $T \in \mathcal{T}$.*

References

[1] M. Atapour, A. Khodkar, and S. Sheikholeslami, *Characterization of double domination subdivision number of trees,* Discrete Applied Mathematics 155 (2007), 1700–1707.

[2] M. Blidia, M. Chellali, and L. Volkmann, *Bounds of the 2-domination number of graphs,* Utilitas Mathematica 71 (2006), 209–216.

[3] M. Blidia, O. Favaron, and R. Lounes, *Locating-domination, 2-domination and independence in trees,* Australasian Journal of Combinatorics 42 (2008), 309–316.

[4] X. Chen and L. Sun, *Some new results on double domination in graphs,* Journal of Mathematical Research and Exposition 25 (2005), 451–456.

[5] J. Fink and M. Jacobson, *n-domination in graphs,* Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, 282–300.

[6] J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita, and L. Volkmann, *Independence and 2-domination in bipartite graphs,* Australasian Journal of Combinatorics 40 (2008), 265–268.

[7] A. Hansberg and L. Volkmann, *On graphs with equal domination and 2-domination numbers,* Discrete Mathematics 308 (2008), 2277–2281.

[8] J. Harant and M. Henning, *A realization algorithm for double domination in graphs,* Utilitas Mathematica 76 (2008), 11–24.

[9] F. Harary and T. Haynes, *Double domination in graphs,* Ars Combinatoria 55 (2000), 201–213.

[10] T. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

[11] T. Haynes, S. Hedetniemi, and P. Slater (eds.), *Domination in Graphs: Advanced Topics,* Marcel Dekker, New York, 1998.

[12] Y. Jiao and H. Yu, *On graphs with equal 2-domination and connected 2-domination numbers,* Mathematica Applicata. Yingyong Shuxue 17 (2004), suppl., 88–92.

[13] M. Krzywkowski, 2-*outer-independent domination in graphs,* submitted.

[14] R. Shaheen, *Bounds for the* 2-*domination number of toroidal grid graphs,* International Journal of Computer Mathematics 86 (2009), 584–588.