Double outer-independent domination in graphs

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Abstract

We initiate the study of double outer-independent domination in graphs. A vertex of a graph is said to dominate itself and all of its neighbors. A double outer-independent dominating set of a graph G is a set D of vertices of G such that every vertex of G is dominated by at least two vertices of D, and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G is the minimum cardinality of a double outer-independent dominating set of G. First we discuss the basic properties of double outer-independent domination in graphs. We find the double outer-independent domination numbers for several classes of graphs. Next we prove lower and upper bounds on the double outer-independent domination number of a graph, and we characterize the extremal graphs. Then we study the influence of removing or adding vertices and edges. We also give Nordhaus-Gaddum type inequalities.
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1 Introduction

Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. The complement of G, denoted by $\bar{G}$, is a graph which has the same vertices as G, and in which two vertices are adjacent if and only if

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they are not adjacent in G. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a vertex is isolated if it has no neighbor, while it is universal if it is adjacent to each other vertex. Let $\delta(G)$ mean the minimum degree among all vertices of G. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. By $K_{p,q}$ we denote a complete bipartite graph with partite sets of cardinalities p and q. By a star we mean the graph $K_{1,m}$, where $m \geq 2$. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. By a subdivided star we mean a graph obtained from a star by subdividing each one of its edges. Generally, let $K_{t_1,t_2,\ldots,t_k}$ denote the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$, where $|S_i| = t_i$ for positive integers $i \leq t$. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. The clique number of G, denoted by $\omega(G)$, is the number of vertices of a greatest complete graph which is a subgraph of G.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of G is dominated by at least one vertex of D, while it is a double dominating set of G if every vertex of G is dominated by at least two vertices of D. The domination (double domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_d(G)$, respectively), is the minimum cardinality of a dominating (double dominating, respectively) set of G. Double domination in graphs was introduced by Harary and Haynes [4], and further studied for example in [1–3, 5, 7, 8]. For a comprehensive survey of domination in graphs, see [6].

A subset $D \subseteq V(G)$ is a double outer-independent dominating set, abbreviated DOIDS, of G if every vertex of G is dominated by at least two vertices of D, and the set $V(G) \setminus D$ is independent. The double outer-independent domination number of a graph G, denoted by $\gamma_d^{oi}(G)$, is the minimum cardinality of a double outer-independent dominating set of G. A double outer-independent dominating set of G of minimum cardinality is called a $\gamma_d^{oi}(G)$-set.

We initiate the study of double outer-independent domination in graphs. First we discuss the basic properties of double outer-independent domination in graphs. We find the double outer-independent domination numbers for several classes of graphs. Next we prove some lower and upper bounds on the double outer-independent domination number of a graph, and we characterize the extremal graphs. Then we study the influence of removing or adding vertices and edges. We also give Nordhaus-Gaddum type inequalities.

2 Results

Since the one-vertex graph, as well as all graphs with an isolated vertex, does not have neither a double outer-independent dominating set nor a double dominating set, in this paper we consider only graphs without isolated vertices.

We begin with the following straightforward observations.

Observation 1 *For every graph G we have $\gamma_d^{oi}(G) \geq \gamma_d(G)$.*

Observation 2 *Every leaf of a graph G is in every DOIDS of G.*

Observation 3 *Every support vertex of a graph G is in every DOIDS of G.*

Observation 4 *If $n \geq 2$ is an integer, then*

(i) $\gamma_d^{oi}(K_n) = \max\{n-1, 2\}$;

(ii) $\gamma_d^{oi}(P_n) = \lfloor (2n+1)/3 \rfloor + 1$.

Let us observe that for any non-negative integer there exists a graph such that the difference between its double outer-independent domination and double domination numbers equals that non-negative integer.

Observation 5 *For every integer $n \geq 3$ we have $\gamma_d^{oi}(K_n) = \gamma_d(K_n) + n - 3$.*

Observation 6 *If $n \geq 3$ is an integer, then $\gamma_d^{oi}(C_n) = \lfloor (2n-1)/3 \rfloor + 1$.*

Observation 7 *For every integer $n \geq 4$ we have $\gamma_d^{oi}(W_n) = \lfloor n/2 \rfloor + 1$.*

Observation 8 *Let p and q be integers such that $p \leq q$. Then*

$$\gamma_d^{oi}(K_{p,q}) = \begin{cases} q+1 & \text{if } p = 1; \\ p+1 & \text{if } p \geq 2. \end{cases}$$

Observation 9 *Let $k \geq 3$ be an integer, and let $t_1, t_2, \dots, t_k$ be positive integers. Then $\gamma_d^{oi}(K_{t_1,t_2,\dots,t_k}) = \sum_{i=1}^{k} t_i - \max\{t_1, t_2, \dots, t_k\}$.*

Observation 10 *For every disjoint graphs $G_1, G_2, \ldots, G_k$ we have $\gamma_d^{oi}(G_1 \cup G_2 \cup \ldots \cup G_k) = \gamma_d^{oi}(G_1) + \gamma_d^{oi}(G_2) + \ldots + \gamma_d^{oi}(G_k)$.*

Since the complement of every double outer-independent dominating set is independent, we get the following lower bound on $\gamma_d^{oi}(G)$ for any graph G.

Observation 11 *For every graph G we have $\gamma_d^{oi}(G) \geq n - \alpha(G)$.*

We have the following lower bound on the double outer-independent domination number of a graph in terms of its clique number.

Fact 12 *For every graph G we have $\gamma_d^{oi}(G) \geq \omega(G) - 1$.*

Proof. Let D be a $\gamma_d^{oi}(G)$-set, and let A be a maximum clique in G. Since $V(G) \setminus D$ is independent, we have $|(V(G) \setminus D) \cap A| \leq 1$. This implies that $|D| \geq |A| - 1$. We now get $\gamma_d^{oi}(G) = |D| \geq |A| - 1 = \omega(G) - 1$. ■

Let us observe that the bound from the previous proposition is tight. For $n \geq 3$ we have $\gamma_d^{oi}(K_n) = n - 1 = \omega(K_n) - 1$.

Now let us observe that for any non-negative integer there exists a graph such that the difference between its double outer-independent domination number and clique number equals that non-negative integer.

Observation 13 *For every positive integer m we have $\gamma_d^{oi}(K_{1,m}) = \omega(K_{1,m}) + m - 1$.*

We now prove that the double outer-independent domination number of a graph is greater than or equal to the minimum degree among all its vertices.

Fact 14 *For every graph G we have $\gamma_d^{oi}(G) \geq \delta(G)$.*

Proof. Let D be any $\gamma_d^{oi}(G)$-set. If $D = V(G)$, then obviously the result is true. Now assume that $D \neq V(G)$. Let x be a vertex which does not belong to D. Since $V(G) \setminus D$ is independent, all neighbors of x belong to the set D. Thus $|D| \geq d_G(x)$. By the definition we have $\delta(G) \leq d_G(x)$. Therefore $\gamma_d^{oi}(G) = |D| \geq d_G(x) \geq \delta(G)$. ■

Since every double outer-independent dominating set has at least two vertices, and all vertices of a graph form a double outer-independent dominating set, we have the following bounds on the double outer-independent domination number of a graph.

Observation 15 *For every graph G we have $2 \leq \gamma_d^{oi}(G) \leq n$.*

We now characterize all graphs which attain the lower bound from the previous observation. For this purpose we introduce a family $\mathcal{G} = \{G_k \colon k \text{ is a non-negative integer}\}$ of graphs, an element of which is given in Figure 1. A graph G_k has $k+2$ vertices.

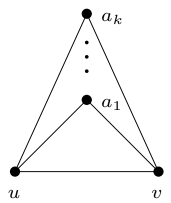

Figure 1: A graph G_k of the family $\mathcal{G}$

Theorem 16 *Let G be a graph. We have $\gamma_d^{oi}(G) = 2$ if and only if $G \in \mathcal{G}$.*

Proof. It is easy to see that $\{u, v\}$ is a DOIDS of any graph of the family $\mathcal{G}$. This implies that $\gamma_d^{oi}(G_k) = 2$, for every graph G_k of the family $\mathcal{G}$.

Now assume that for some graph G we have $\gamma_d^{oi}(G) = 2$. Let D be a $\gamma_d^{oi}(G)$-set. The vertices of D we denote by u and v. Since D is a double dominating set, we conclude that the vertices u and v are adjacent, and every vertex of $V(G) \setminus D$ is adjacent to both vertices u and v. Moreover, there is no edge between any two vertices of $V(G) \setminus D$ as the set D is outer-independent. It is easy to observe that G is a graph of the form presented in Figure 1. ■

We now characterize the graphs attaining the upper bound from Observation 15, that is, the graphs with double outer-independent domination number equaling the number of vertices.

Theorem 17 *Let G be a graph. Then $\gamma_d^{oi}(G) = n$ if and only if every vertex of G is a leaf or a support vertex.*

Proof. The sufficiency is true by Observations 1 and 2. Now assume that some vertex of a graph G, say x, is neither a leaf nor a support vertex. Thus x has at least two neighbors. Moreover, each of these neighbors has a neighbor different from x since x is not a support vertex. It is not difficult to observe that $V(G) \setminus \{x\}$ is a DOIDS of the graph G. Therefore $\gamma_d^{oi}(G) \leq n - 1 < n$. ■

Corollary 18 *Let G be a graph. If some vertex of G is neither a leaf nor a support vertex, then $\gamma_d^{oi}(G) \leq n - 1$.*

We now characterize the graphs attaining the bound from the previous corollary.

Theorem 19 *Let G be a graph. We have $\gamma_d^{oi}(G) = n-1$ if and only if at least one vertex of G is neither a leaf nor a support vertex, and the subgraph of G induced by the vertices which are neither leaves nor support vertices is a complete graph or a path on three vertices such that the central vertex has exactly two neighbors in the graph G.*

Proof. Let G be a graph such that its subgraph induced by the vertices which are neither leaves nor support vertices is a complete graph or a path on three vertices such that the central vertex has exactly two neighbors in the graph G. First assume that it is a path P_3, say abc. It is not difficult to observe that $V(G) \setminus \{b\}$ is a DOIDS of the graph G. Thus $\gamma_d^{oi}(G) \leq n-1$. Now let D be any $\gamma_d^{oi}(G)$-set. By Observations 2 and 3, all leaves and support vertices belong to the set D. Moreover, the vertex b has to be dominated twice, thus at least two of the vertices a, b and c belong to the set D. Therefore $\gamma_d^{oi}(G) \geq n-1$. Now assume that the subgraph of G induced by the vertices which are neither leaves nor support vertices is a complete graph. Let x be any vertex of this subgraph. Let us observe that $V(G) \setminus \{x\}$ is a DOIDS of the graph G. Thus $\gamma_d^{oi}(G) \leq n-1$. Now let D be any $\gamma_d^{oi}(G)$-set. By Observations 2 and 3, all leaves and support vertices belong to the set D. Moreover, since $V(G) \setminus D$ is independent, at most one of the remaining vertices does not belong to the set D. Therefore $\gamma_d^{oi}(G) \geq n-1$. We now conclude that $\gamma_d^{oi}(G) = n-1$.

Now assume that for some graph G we have $\gamma_d^{oi}(G) = n-1$. Suppose that the subgraph of G induced by the vertices which are neither leaves nor support vertices, say H, is neither a complete graph nor a path on three vertices such that the central vertex has exactly two neighbors in the graph G. Thus $|V(H)| \geq 2$. Let us observe that there exist two nonadjacent vertices of H, say x and y, such that no common neighbor of x and y has degree two in the graph G. It is not very difficult to observe that $V(G) \setminus \{x, y\}$ is a DOIDS of the graph G. Therefore $\gamma_d^{oi}(G) \leq n-2 < n-1$, a contradiction. ■

We have the following upper bound on the double outer-independent domination number of a tree in terms of its independence number and the number of support vertices.

Theorem 20 *For every tree T of order at least three with s support vertices we have $\gamma_d^{oi}(T) \leq \alpha(T) + s$.*

Proof. Let n mean the number of vertices of the tree T. We proceed by induction on this number. If $\mathrm{diam}(T) = 1$, then $T = P_2$. We have $\gamma_d^{oi}(P_2) = 2 < 1+2 = \alpha(P_2)+s$. Now assume that $\mathrm{diam}(T) = 2$. Thus T is a star. We have $\gamma_d^{oi}(T) = n = n-1+1 = \alpha(T)+s$. Now let us assume that

$\mathrm{diam}(T) = 3$. Thus T is a double star. We have $\gamma_d^{oi}(T) = n = n - 2 + 2$ $= \alpha(T) + s$.

Now assume that $\mathrm{diam}(T) \geq 4$. Thus the order n of the tree T is at least five. We obtain the result by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $s' = s$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observation 3 we have $x \in D'$. It is easy to see that $D' \cup \{y\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$. Let us observe that there exists a maximum independent set of T' that does not contain the vertex x. Let A' be such a set. It is easy to observe that $D' \cup \{y\}$ is an independent set of the tree T. Thus $\alpha(T) \geq \alpha(T') + 1$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1 \leq \alpha(T') + s' + 1 = \alpha(T') + s + 1 \leq \alpha(T) + s$. Henceforth, we can assume that all support vertices of T are weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, and w be the parent of u in the rooted tree. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that $d_T(u) \geq 3$. Let $T' = T - T_v$. We have $s' = s - 1$. Let D' be any $\gamma_d^{oi}(T')$-set. Obviously, $D' \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$. Now let A' be a maximum independent set of T'. It is easy to see that $D' \cup \{t\}$ is an independent set of the tree T. Thus $\alpha(T) \geq \alpha(T') + 1$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq \alpha(T') + s' + 2$ $\leq \alpha(T) + s$.

Now assume that $d_T(u) = 2$. First assume that there is a child of w other than u, say k, such that the distance of w to the most distant vertex of T_k is one or three. It suffices to consider only the possibilities when T_k is a path P_3, or k is a leaf. Let $T' = T - T_u$. We have $s' = s - 1$. Let us observe that there exists a $\gamma_d^{oi}(T')$-set that contains the vertex w. Let D' be such a set. It is easy to observe that $D' \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2$. Now let A' be a maximum independent set of the tree T'. Obviously, $D' \cup \{t\}$ is an independent set of T. Thus $\alpha(T) \geq \alpha(T') + 1$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 2 \leq \alpha(T') + s' + 2$ $\leq \alpha(T) + s$.

Now assume that for every child of w other than u, say k, the distance of w to the most distant vertex of T_k is two. It suffices to consider only the possibility when k is a support vertex of degree two. Let $T' = T - T_v$. We have $s' = s$. Let D' be any $\gamma_d^{oi}(T')$-set. By Observations 2 and 3 we have $u, k, w \in D'$. Let us observe that $D' \setminus \{u\} \cup \{v, t\}$ is a DOIDS of the tree T. Thus $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1$. Now let A' be a maximum independent set of T'. It is easy to see that $D' \cup \{t\}$ is an independent set of the tree T. Thus $\alpha(T) \geq \alpha(T') + 1$. We now get $\gamma_d^{oi}(T) \leq \gamma_d^{oi}(T') + 1 \leq \alpha(T') + s' + 1$ $\leq \alpha(T) + s$. ■

We have the following bounds on the double outer-independent domination number of a graph in terms of its order and size.

Proposition 21 *For every graph G we have*

$$\frac{2n-3-\sqrt{(2n-3)^2-8(m-1)}}{2} \leq \gamma_d^{oi}(G) \leq \frac{2n-3+\sqrt{(2n-3)^2-8(m-1)}}{2}.$$

Proof. Let D be a $\gamma_d^{oi}(G)$-set. Let t denote the number of edges between the vertices of D and the vertices of $V(G) \setminus D$. We have $m \leq t + |E(G[D])|$. Obviously, $t \leq |D| \cdot |V(G) \setminus D|$. Notice that $|E(G[D])| \leq (|D|-1)(|D|-2)/2$. Now simple calculations imply the result. ■

We now study the influence of the removal of a vertex of a graph on its double outer-independent domination number.

Proposition 22 *Let G be a graph, and let v be a vertex of G. Assume that $G-v$ has no isolated vertex. Then $\gamma_d^{oi}(G) - 2 \leq \gamma_d^{oi}(G-v) \leq \gamma_d^{oi}(G)$ $+ d_G(v) - 1$.*

Proof. Let D be any $\gamma_d^{oi}(G)$-set. If $v \notin D$, then it is easy to see that D is a DOIDS of the graph $G-v$. Now assume that $v \in D$. Let $v_1, v_2, \ldots, v_{d_G(v)}$ be the neighbors of v. Let $i \in \{1, 2, \ldots, d_G(v)\}$. Let x_i be a neighbor of v_i different from v. If $v_i \in D$, then let u_i mean x_i, otherwise let it mean v_i. Let us observe that $D \cup \{u_1, u_2, \ldots, u_{d_G(v)}\} \setminus \{v\}$ is a DOIDS of the graph G. Thus $\gamma_d^{oi}(G-v) \leq |D \cup \{u_1, u_2, \ldots, u_{d_G(v)}\} \setminus \{v\}| \leq |D| + d_G(v) - 1$ $= \gamma_d^{oi}(G) + d_G(v) - 1$. Now let D' be any $\gamma_d^{oi}(G-v)$-set. If some vertex of $N_G(v)$ belongs to the set D', then it is easy to see that $D' \cup \{v\}$ is a DOIDS of the graph G. Now assume that no vertex of $N_G(v)$ belongs to the set D'. Let x be any neighbor of v in G. It is easy to observe that $D' \cup \{v, x\}$ is a DOIDS of the graph G. Therefore $\gamma_d^{oi}(G) \leq \gamma_d^{oi}(G-v) + 2$. ■

Let us observe that the bounds from the previous proposition are tight. For the lower bound, consider the graphs G_k of the form presented in Figure 2. We have $\gamma_d^{oi}(G_k) = 12k = 12k - 2 + 2 = \gamma_d^{oi}(G_k - v_i) + 2$. For the upper bound, let G be a graph obtained from a star $K_{1,m}$ by subdividing every edge thrice. The vertex of minimum eccentricity we denote by x. We have $\gamma_d^{oi}(G-x) = \gamma_d^{oi}(mP_4) = m \cdot \gamma_d^{oi}(P_4) = 4m = 3m + 1 + m - 1$ $= \gamma_d^{oi}(G) + d_G(x) - 1$.

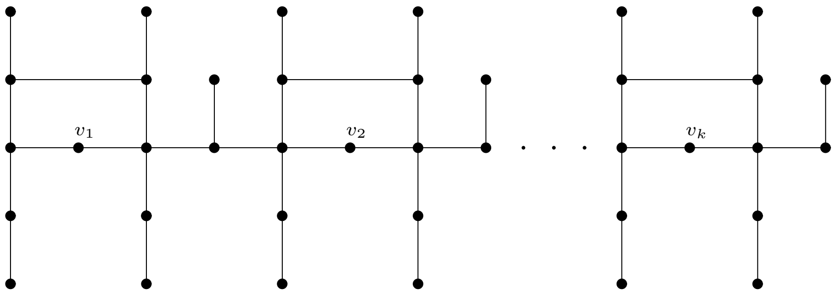

Figure 2: A graph G_k having $13k$ vertices

We now show that for any non-negative integer there exists a graph such that the removal of some its vertex increases the double outer-independent domination number by that non-negative integer.

Fact 23 *For every non-negative integer k there exists a graph G such that for some vertex v of G we have $\gamma_d^{oi}(G-v) - \gamma_d^{oi}(G) = k$.*

Proof. Let $m = k + 2$. Let G be a graph obtained from a star $K_{1,m}$ by subdividing every edge twice. The vertex of minimum eccentricity we denote by x. It is easy to see that $\gamma_d^{oi}(G) = 2m+2$. We have $G-x = mP_3$. We now get $\gamma_d^{oi}(G-x) - \gamma_d^{oi}(G) = \gamma_d^{oi}(mP_3) - \gamma_d^{oi}(K_{1,m}) = 3m - (2m+2)$ $= m - 2 = k$. ∎

We now study the influence of the removal of an edge of a graph on its double outer-independent domination number.

Proposition 24 *Let G be a graph. For every edge e of G we have*

$$\gamma_d^{oi}(G-e) \in \{\gamma_d^{oi}(G) - 1, \gamma_d^{oi}(G), \gamma_d^{oi}(G) + 1\}.$$

Proof. Let D be a $\gamma_d^{oi}(G)$-set, and let $e = xy$ be an edge of G. Since the set $V(G) \setminus D$ is independent, some of the vertices x and y belongs to the set D. Without loss of generality we may assume that $x \in D$. If $y \in D$, then it is easy to see that D is a DOIDS of the graph $G-e$. If $y \notin D$, then $D \cup \{y\}$ is a DOIDS of $G-e$. Thus $\gamma_d^{oi}(G-e) \le \gamma_d^{oi}(G) + 1$. Now let D' be a $\gamma_d^{oi}(G-e)$-set. If some of the vertices x and y belongs to the set D', then D' is a DOIDS of the graph G. If none of the vertices x and y belongs to the set D', then it is easy to observe that $D' \cup \{x\}$ is a DOIDS of the graph G. Therefore $\gamma_d^{oi}(G) \le \gamma_d^{oi}(G-e) + 1$. ∎

Let us observe that the bounds following from the previous proposition are tight. For the lower bound, let us remove an edge of the complete graph K_4. For the upper bound, consider a path P_6 and its central edge.

Similarly, adding an edge has the following influence on the double outer-independent domination number of a graph.

Proposition 25 *Let G be a graph. If $e \notin E(G)$, then*

$$\gamma_d^{oi}(G+e) \in \{\gamma_d^{oi}(G)-1, \gamma_d^{oi}(G), \gamma_d^{oi}(G)+1\}.$$

We now give Nordhaus-Gaddum type inequalities for the sum of the double outer-independent domination number of a graph and its complement.

Theorem 26 *For every graph G we have $n-1 \le \gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) \le 2n$.*

Proof. Let D be any $\gamma_d^{oi}(G)$-set. Since $V(G) \setminus D$ is independent, the vertices of $V(G) \setminus D$ form a clique in $\bar{G}$. Let $\bar{D}$ be any $\gamma_d^{oi}(\bar{G})$-set. The vertices of $V(G) \setminus D$ form a clique in $\bar{G}$, thus at most one of them does not belong to $\bar{D}$ as $V(\bar{G}) \setminus \bar{D}$ is independent. Therefore $|\bar{D}| \ge |V(G) \setminus D| - 1$. We now get $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) \ge |D| + |V(G) \setminus D| - 1 = n-1$.

Obviously, $\gamma_d^{oi}(G) \le n$ and $\gamma_d^{oi}(\bar{G}) \le n$. Thus $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) \le 2n$. ∎

Now let us observe that the bounds from the previous theorem are tight. For the lower bound, consider the graphs H and $\bar{H}$ given in Figure 3. Let us observe that H and $\bar{H}$ are isomorphic. It is not difficult to see that $\{v_1, v_2, v_3, v_4\}$ is a DOIDS of the graph H, and $\{u_2, u_3, u_4, u_5\}$ is a DOIDS of the graph $\bar{H}$. Therefore $\gamma_d^{oi}(H) + \gamma_d^{oi}(\bar{H}) \le |\{v_1, v_2, v_3, v_4\}|$ $+|\{u_2, u_3, u_4, u_5\}| = 8 = n-1$. On the other hand, by Theorem 26 we have $\gamma_d^{oi}(H) + \gamma_d^{oi}(\bar{H}) \ge n-1$. Therefore $\gamma_d^{oi}(H) + \gamma_d^{oi}(\bar{H}) = n-1$. For the upper bound, consider the path P_4 and its complement $\bar{P_4} = P_4$. We have $\gamma_d^{oi}(P_4) + \gamma_d^{oi}(\bar{P_4}) = 2 \cdot \gamma_d^{oi}(P_4) = 8 = 2n$.

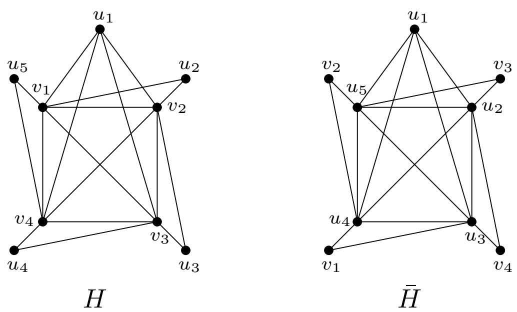

Figure 3: The graphs H and $\bar{H}$

We now prove that the path on four vertices and its complement (which are isomorphic) are the only graphs which attain the upper bound from Theorem 26.

Theorem 27 *Let G be a graph. We have $\gamma_d^{oi}(G)+\gamma_d^{oi}(\bar{G}) = 2n$ if and only if $G = \bar{G} = P_4$.*

Proof. Obviously, $\gamma_d^{oi}(P_4) + \gamma_d^{oi}(\bar{P}_4) = 2 \cdot \gamma_d^{oi}(P_4) = 2n$. Now assume that for some graph G we have $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) = 2n$. This implies that $\gamma_d^{oi}(G) = n$ and $\gamma_d^{oi}(\bar{G}) = n$. By Theorem 17, every vertex of G is a leaf or a support vertex, and every vertex of $\bar{G}$ is a leaf or a support vertex. Suppose that some vertex, say x, is simultaneously a leaf and a support vertex of one of the graphs G or $\bar{G}$, say G. This implies that the component of G which contains the vertex x is a complete graph on two vertices. The neighbor of x we denote by y. If $G = K_2$, then $\bar{G}$ has an isolated vertex, a contradiction. Now assume that $G \neq K_2$. Since every component of G has at least two vertices, we have $n \geq 4$. In the graph $\bar{G}$ the vertex x is not a leaf as it has $n - 2$ neighbors and $n - 2 \geq 2 > 1$. Thus x must be a support vertex of $\bar{G}$. But each neighbor of x in $\bar{G}$ is also adjacent to y, so x is not adjacent to any leaf, and it is not a support vertex, a contradiction. We now conclude that no vertex is simultaneously a leaf and a support vertex neither of G nor of $\bar{G}$. Suppose that some vertex, say x, is a leaf of both graphs G and $\bar{G}$. Since in each one of the graphs G and $\bar{G}$ the vertex x has exactly one neighbor, the graph G (as well as the graph $\bar{G}$) has three vertices. It is not difficult to verify that then G or $\bar{G}$ has an isolated vertex, and we do not consider such graphs. Therefore every leaf of G is a support vertex of $\bar{G}$, and every leaf of $\bar{G}$ is a support vertex of G. Let us observe that for every graph, the number of leaves is greater than or equal to the number of support vertices. This implies that exactly half of all vertices are leaves of G (and they are support vertices of $\bar{G}$), and the remaining half of vertices are support vertices of G (and they are leaves of $\bar{G}$). Consequently, every support vertex is weak. Moreover, the number $n = |V(G)| = |V(\bar{G})|$ is even. Suppose that $n \geq 6$. Thus in G there are at least three leaves, say a, b and c. The support vertex adjacent to a we denote by x. Since every support vertex is weak, the vertex x is not adjacent to any one of the vertices b and c. Thus x (which is a leaf in $\bar{G}$) is adjacent to both vertices b and c in $\bar{G}$, a contradiction as leaf has only one neighbor. This implies that $n = 4$. Since exactly half of all vertices are leaves and half are support vertices, it is not very difficult to get that $G = P_4$. Obviously, $\bar{G} = \bar{P}_4 = P_4$. ■

Corollary 28 *If G is a graph different from P_4, then $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) \leq 2n - 1$.*

We now characterize all graphs, which attain the bound from the previous corollary. For this purpose we introduce a family $\mathcal{H} = \{H_k \colon k \geq 3 \text{ is an integer}\}$ of graphs that can be obtained from a complete graph on $k \geq 3$ vertices by attaching a path on two vertices by joining one of its vertices to all but one vertex of that complete graph. In Figure 4 we present a few graphs of the family $\mathcal{H}$.

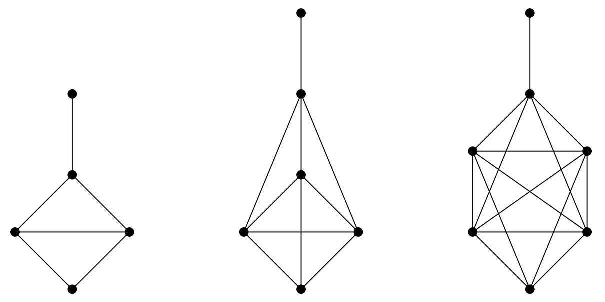

Figure 4: The graphs H_3, H_4 and H_5 of the family $\mathcal{H}$

Theorem 29 *Let G be a graph. We have $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) = 2n - 1$ if and only if $G \in \mathcal{H}$ or $\bar{G} \in \mathcal{H}$.*

Proof. Assume that a graph G or its complement $\bar{G}$ belongs to the family $\mathcal{H}$. Without loss of generality we assume that $G \in \mathcal{H}$. By Theorem 19 we have $\gamma_d^{oi}(G) = n - 1$. The leaf of G we denote by x, and its neighbor we denote by y. Let z be the vertex of G which is not adjacent to y. It is easy to see that all vertices excluding x and z are leaves of $\bar{G}$. The vertex z is adjacent to y in $\bar{G}$, thus z is a support vertex of $\bar{G}$. The vertex x is also a support vertex of $\bar{G}$, as it is adjacent to the remaining leaves in $\bar{G}$. By Observations 2 and 3, all leaves and support vertices belong to every double outer-independent dominating set. Thus $\gamma_d^{oi}(\bar{G}) = n$. We now get $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) = n - 1 + n = 2n - 1$.

Now assume that for some graph G we have $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) = 2n - 1$. This implies that $\gamma_d^{oi}(G) = n - 1$ or $\gamma_d^{oi}(\bar{G}) = n - 1$. Without loss of generality we assume that $\gamma_d^{oi}(G) = n - 1$. Consequently, $\gamma_d^{oi}(\bar{G}) = n$. By Theorem 17, every vertex of $\bar{G}$ is a leaf or a support vertex. Suppose that some vertex, say x, is a leaf of both graphs G and $\bar{G}$. Thus $n = 3$, and G or $\bar{G}$ has an isolated vertex, a contradiction. Therefore no vertex is at the same time a leaf of both graphs G and $\bar{G}$. Now suppose that some component of G is a complete graph on two vertices. Let a denote a vertex of such component. Since a is a leaf, and no vertex is a leaf of both graphs G

and $\bar{G}$, we conclude that a is not a leaf of $\bar{G}$. Therefore it is a support vertex of $\bar{G}$. This implies that some vertex of G different from a is adjacent to all vertices of G excluding a, a contradiction. We conclude that no component of G is a complete graph on two vertices. Now suppose that G has at least three support vertices, say a, b and c. Let d (e, f, respectively) denote a leaf adjacent to a (b, c, respectively). Since every vertex of $\bar{G}$ is a leaf or a support vertex, the graph $\bar{G}$ has some leaf, say x. Thus the vertex x has $n-2$ neighbors in G, a contradiction as every vertex of G is adjacent to at most one of the vertices d, e and f. Therefore G has at most two support vertices. Now suppose that some support vertex of G, say x, is strong. Let a denote a leaf adjacent to x. Since a is a leaf, and no vertex is a leaf of both graphs G and $\bar{G}$, we conclude that a is not a leaf of $\bar{G}$. Therefore it is a support vertex of $\bar{G}$. This implies that some vertex of G different from a is adjacent to all vertices of G excluding a. This is a contradiction as every vertex of G is adjacent to a if and only if it is adjacent to b. Therefore every support vertex of G is weak. First assume that G has exactly two support vertices. Thus G has exactly two leaves, say x and y. Since no vertex is a leaf of both graphs G and $\bar{G}$, the vertices x and y are support vertices of $\bar{G}$. Let a denote a leaf adjacent to x in $\bar{G}$, and let b denote a leaf adjacent to y in $\bar{G}$. Thus a (b, respectively) is adjacent to every vertex in G excluding x (y, respectively). Therefore in G the vertex x is adjacent to b, and y is adjacent to a. Now let us consider a vertex of G, say u, which is neither a support vertex nor a leaf. In the graph $\bar{G}$, the vertex u is a leaf or a support vertex. Since u is adjacent to none of the vertices x and y in G, it is adjacent to both these vertices in $\bar{G}$. Thus $d_{\bar{G}}(u) \geq 2$, and consequently, u is not a leaf of $\bar{G}$. Therefore u is a support vertex of $\bar{G}$. This is a contradiction as a and b are the only leaves of $\bar{G}$, and x and y are the only support vertices of $\bar{G}$. Now assume that G has exactly one support vertex, say x. The leaf adjacent to x we denote by y. Since $\gamma_d^{oi}(G) = n-1$, by Theorem 19, the subgraph of G, say H, induced by the vertices which are neither leaves nor support vertices is a path on three vertices or a complete graph. If H contains only one vertex, then $G = P_3$. Thus every vertex of G is a leaf or a support vertex, a contradiction. Now assume that $H = K_2$. This implies that every vertex of G is a leaf or a support vertex, or G has a universal vertex, that is, $\bar{G}$ has an isolated vertex, a contradiction. Now assume that H is a path P_3, say abc. Since none of the vertices a, b and c is a leaf or a support vertex, the vertices a and c are adjacent to the vertex x. We have $bx \notin E(G)$, otherwise x is an isolated vertex of $\bar{G}$. It is not difficult to verify that $\gamma_d^{oi}(\bar{G}) = 4$. We get $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) = 4 + 4 = 2n - 2 < 2n - 1$, a contradiction. Now assume that H is a complete graph on at least three vertices. Suppose that at least two vertices of H, say a and b, are not adjacent to x in G. Therefore x is adjacent to both a and b in the graph $\bar{G}$. Thus x is not a leaf, and consequently, it is a support vertex of $\bar{G}$. Consequently, some vertex of G is adjacent to all other vertices excluding a. This is a contradiction as no vertex is adjacent in G to both vertices a and y, and the vertices a and y are not adjacent. We now conclude that in G the vertex x is adjacent to at least $|V(H)|-1$ vertices of H. Let us observe that it cannot be adjacent to every vertex of H, otherwise x is an isolated vertex of $\bar{G}$, a contradiction. Therefore in G the vertex x is adjacent to all but one vertex of H. We now conclude that $G \in \mathcal{H}$. ■

Corollary 30 *If $G \neq P_4$, $G \notin \mathcal{H}$ and $\bar{G} \notin \mathcal{H}$, then $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) \leq 2n-2$.*

We now improve the lower bound from Theorem 26.

Theorem 31 *For every graph G with l leaves we have $\gamma_d^{oi}(G) + \gamma_d^{oi}(\overline{G}) \geq n+l-2$.*

Proof. Let D and $\bar{D}$ be any $\gamma_d^{oi}(G)$-set and $\gamma_d^{oi}(\bar{G})$-set, respectively. If some leaf of G does not belong to the set $\bar{D}$, then $\gamma_d^{oi}(\bar{G}) \geq n-2$, and we easily obtain the result. Now assume that all leaves of G belong to the set $\bar{D}$. Since $V(G) \setminus D$ is independent, the vertices of $V(G) \setminus D$ form a clique in $\bar{G}$. Consequently, at most one of them does not belong to $\bar{D}$ as $V(\bar{G}) \setminus \bar{D}$ is independent. Therefore $|\bar{D}| \geq |V(G) \setminus D| + l - 1$. We now get $\gamma_d^{oi}(G) + \gamma_d^{oi}(\bar{G}) \geq |D| + |V(G) \setminus D| + l - 1 > n + l - 2$. ■

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