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An alternative proof of a lower bound on the 2-domination number of a tree

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Abstract

A 2-dominating set of a graph G is a set D of vertices of G such that every vertex not in D has a at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. Fink and Jacobson [*n-domination in graphs,* Graph theory with applications to algorithms and computer science, Wiley, New York, 1985, $283-300$] established the following lower bound on the 2-domination number of a tree in term of its order, $\gamma_2(T) \geq (n+1)/2$. We give an alternative proof of this bound.

Keywords: 2-domination, tree.

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Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. If $X \subseteq V(G)$, then let $N_G(X) = \bigcup_{v \in X} N_G(v)$. For $Y \subseteq V(G)$ we define $N_Y(v) = N_G(v) \cap Y$ and $N_Y(X) = N_G(X) \cap Y$.

A 2-dominating set of a graph G is a set D of vertices of G such that every vertex not in D has a at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. The concept of 2-domination was introduced by Fink and Jacobson [1, 2]. They [1] established the following lower bound on the 2-domination number of a tree in term of its order, $\gamma_2(T) \geq (n+1)/2$. We give an alternative proof of this bound.

Theorem ([1]) *For every tree T of order n we have $\gamma_2(T) \geq (n+1)/2$.*

Proof. First we prove that if $D \subseteq V(T)$ is a 2-dominating set of T, then for every $S \subseteq V(T) - D$ we have $|N_D(S)| > |S|$. We prove this by the induction on the cardinality of S. By the definition of a 2-dominating set, every 1-element subset of $V(T) - D$ has at least two neighbors in D. Let k be an integer such that $k \geq 2$. Assume that every k'-element subset of $V(T) - D$ has at least $k'+1$ neighbors in D, for every positive integer $k' < k$. Let

$A = \{v_1, v_2, \ldots, v_k\} \subseteq V(T) - D$. By the inductive hypothesis we have $|N_D(A - v_k)| \geq k$. If $|N_D(A-v_k)| \geq k+1$, then $|N_D(A)| \geq k+1$. Now assume that $|N_D(A-v_k)| = k$. Of course, $|N_D(A)| \geq k$. Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be the numbers of vertices of $A - v_k$ in particular connected components of $\langle (A - v_k) \cup N_D(A - v_k) \rangle$. By inductive hypothesis we get $|N_D(A - v_k)| \geq \alpha_1 + 1 + \alpha_2 + 1 + \ldots + \alpha_t + 1 = k + t - 1$. On the other hand, $|N_D(A - v_k)| = k$. This implies that $t = 1$, that is, $\langle (A - v_k) \cup N_D(A - v_k) \rangle$ is connected. Suppose that $|N_D(A)| = k$. This implies that $\{x, y\} \subseteq N_D(v_k) \subseteq N_D(A - v_k)$, for some $x, y \in D$. Since there are two distinguish paths between x and y, there is a cycle, a contradiction. Thus $|N_D(A)| \geq k + 1$. Considering the expression $|N_D(S)| > |S|$ for $S = V(T) - D$ we get $|D| > |V(T)|/2$. Therefore $\gamma_2(T) \geq (n + 1)/2$. $\square$

References

[1] J. Fink and M. Jacobson, *n-domination in graphs,* Graph theory with applications to algorithms and computer science, Wiley, New York, 1985, 282−300.

[2] J. Fink and M. Jacobson, *On n-domination, n-dependence and forbidden subgraphs,* Graph theory with applications to algorithms and computer science, Wiley, New York, 1985, 301−311.

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